# Relating Different Quantum Generalizations of the Conditional Rényi Entropy

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- Rényi Divergences and Entropies play a fundamental role in information theory, for example when studying error exponents (Gallager) or the strong converse (Arimoto).

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- Classically, the Rényi divergence is directly related to the cumulant generating function of the log-likelihood ratio.
- In quantum theory, this connection is not so direct, and the quantum generalization is thus also not unique.
- Here we study different quantum generalizations of the conditional Rényi entropy, discuss some of their properties and find new relations between them.

# Rényi Divergence

- Traditional quantum generalization via quantum $f$-divergences or quasi-entropies (see, e.g., Ohya and Petz, OP'93):

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \operatorname{tr} \left\{ \rho^\alpha \sigma^{1-\alpha} \right\}.$$

- Recently, an alternative quantum generalization has been proposed (MDSFT'13, WWY'14):

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \operatorname{tr} \left\{ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right\}.$$

# Conditional Rényi Entropy

- The conditional von Neumann entropy can be written as

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\begin{aligned}
H(A|B)_\rho &:= -D(\rho_{AB} \| 1_A \otimes \rho_B) \\
&= \sup_{\sigma_B} -D(\rho_{AB} \| 1_A \otimes \sigma_B)
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- For the old Rényi relative entropy:

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H_\alpha^{\downarrow}(A|B)_\rho &:= -D_\alpha(\rho_{AB} \| 1_A \otimes \rho_B), \\
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- For the new Rényi divergence:

$$\begin{aligned} \widetilde{H}_\alpha^\downarrow(A|B)_\rho &:= -\widetilde{D}_\alpha(\rho_{AB} \| 1_A \otimes \rho_B), \\ \widetilde{H}_\alpha^\uparrow(A|B)_\rho &:= \sup_{\sigma_B} -\widetilde{D}_\alpha(\rho_{AB} \| 1_A \otimes \sigma_B). \end{aligned}$$

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  $$H_{\min}(A|B)_\rho \equiv \widetilde{H}^{\uparrow}_{\infty}(A|B)_\rho = \sup\left\{\lambda \in \mathbb{R} : \rho_{AB} \leq 2^{-\lambda} 1_A \otimes \sigma_B\right\}.$$

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- $\alpha = 2$: We find the collision entropy (Renner'05):

$$\widetilde{H}^{\downarrow}_{2}(A|B)_\rho = -\log \operatorname{tr}\left\{\left(\rho_{AB}\left(1_A \otimes \rho_B^{-\frac{1}{2}}\right)\right)^2\right\}.$$

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- $\alpha = \frac{1}{2}$: This yields the conditional max-entropy (KRS'09),

$$H_{\max}(A|B)_\rho \equiv \widetilde{H}^{\uparrow}_{1/2}(A|B)_\rho = \sup_{\sigma_B} \log F\left(\rho_{AB}, 1_A \otimes \sigma_B\right),$$

where $F(\cdot,\cdot)$ denotes Uhlmann's fidelity.

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where $F(\cdot,\cdot)$ denotes Uhlmann's fidelity.

- $\alpha = 0$: We find the Hartley entropy (Renner'05),

$$H^{\uparrow}_{0}(A|B)_\rho = \sup_{\sigma_B} \log \operatorname{tr}\{\Pi_{\rho_{AB}} 1_A \otimes \sigma_B\},$$

where $\Pi_\rho$ denotes the projector onto the support of $\rho$.

# Properties of Conditional Rényi Entropy

- Monotonicity in $\alpha$: If $\alpha \geq \beta$ then $H_\alpha(A|B)_\rho \leq H_\beta(A|B)_\rho$ for all four variants of the conditional entropy.

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- Data-Processing: For any quantum channel $\Lambda_{B \to B'}$, we have

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\begin{aligned}
H_\alpha(A|B)_\rho &\leq H_\alpha(A|B')_{\Lambda(\rho)} \qquad \text{for} \qquad \alpha \in [0,2], \\
\widetilde{H}_\alpha(A|B)_\rho &\leq \widetilde{H}_\alpha(A|B')_{\Lambda(\rho)} \qquad \text{for} \qquad \alpha \in \left[\frac{1}{2}, \infty\right].
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  This holds for both the $\uparrow$ and $\downarrow$ variants.
- Ariki-Lieb-Thirring trace inequalities: We have

$$H_\alpha(A|B)_\rho \geq \widetilde{H}_\alpha(A|B)_\rho$$

  for both the $\uparrow$ and $\downarrow$ variants.

# Alternative Representation

$$H_\alpha^\uparrow(A|B)_\rho := \sup_{\sigma_B} -D_\alpha(\rho_{AB} \| 1_A \otimes \sigma_B).$$

- Using a quantum Hölder inequality, we find the state $\sigma_B$ that optimizes $H_\alpha^\uparrow(A|B)_\rho$ above.
- (Alternatively, a quantum generalization of Sibson's identity (SW'13) can be used for this purpose.)

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- (Alternatively, a quantum generalization of Sibson's identity (SW'13) can be used for this purpose.)
- This yields the alternative expression

$$H_\alpha^{\uparrow}(A|B)_\rho = \frac{\alpha}{1-\alpha} \log \operatorname{tr}\left\{ \left( \operatorname{tr}_A\{\rho_{AB}^\alpha\} \right)^{\frac{1}{\alpha}} \right\}.$$

# Duality Relations

- For any tripartite pure state $\rho_{ABC}$, we have the *duality relation*

$$H(A|B)_\rho + H(A|C)_\rho = 0\,.$$

- Proof: Simply write $H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B)$ and $H(A|C)_\rho = H(\rho_{AC}) - H(\rho_C)$ and note that the spectra of $\rho_{AB}$ and $\rho_C$ as well as the spectra of $\rho_B$ and $\rho_{AC}$ agree.

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- The significance of this relation is manifold — for example it turns out to be useful in cryptography where the entropy of an adversarial party, $C$, can be estimated using local state tomography by two honest parties, $A$ and $B$.

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- The significance of this relation is manifold — for example it turns out to be useful in cryptography where the entropy of an adversarial party, $C$, can be estimated using local state tomography by two honest parties, $A$ and $B$.
- We want to see if such relations hold more generally for conditional Rényi entropies.

# Main Result

- Recall the definitions:

$$H_\alpha^{\uparrow}(A|B)_\rho := \sup_{\sigma_B} -D_\alpha(\rho_{AB} \| 1_A \otimes \sigma_B),$$

$$\widetilde{H}_\alpha^{\downarrow}(A|B)_\rho := -\widetilde{D}_\alpha(\rho_{AB} \| 1_A \otimes \rho_B).$$

- Surprisingly, these definitions are related via the following:

**Duality Relation**

Let $\alpha, \beta \in (0,1) \cup (1,\infty)$ with $\alpha \cdot \beta = 1$ and $\rho_{ABC}$ be pure. Then,

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- The relation $\alpha \cdot \beta = 1$ ensures that $[0,2]$ gets mapped to $[\frac{1}{2},\infty]$. This means that the entropy satisfies data-processing if and only if its dual does too.

# Proof

- Substituting $\beta = \frac{1}{\alpha}$ and employing the alternative representation, it remains to show that

$$H_\alpha^\uparrow(A|B)_\rho = \frac{\alpha}{1-\alpha} \log \operatorname{tr} \left\{ \left( \operatorname{tr}_A\{\rho_{AB}^\alpha\} \right)^{\frac{1}{\alpha}} \right\}$$

is equal to

$$-\widetilde{H}_\beta^\downarrow(A|C)_\rho = -\frac{1}{1-\beta} \log \operatorname{tr} \left\{ \left( \left( 1_A \otimes \rho_C^{\frac{1-\beta}{2\beta}} \right) \rho_{AC} \left( 1_A \otimes \rho_C^{\frac{1-\beta}{2\beta}} \right) \right)^\beta \right\}$$
$$= \frac{\alpha}{1-\alpha} \log \operatorname{tr} \left\{ \left( \left( 1_A \otimes \rho_C^{\frac{\alpha-1}{2}} \right) \rho_{AC} \left( 1_A \otimes \rho_C^{\frac{\alpha-1}{2}} \right) \right)^{\frac{1}{\alpha}} \right\},$$

- It is sufficient to show that the operators

$$\operatorname{tr}_A\{\rho_{AB}^\alpha\} \qquad \text{and} \qquad \left( 1_A \otimes \rho_C^{\frac{\alpha-1}{2}} \right) \rho_{AC} \left( 1_A \otimes \rho_C^{\frac{\alpha-1}{2}} \right)$$

are unitarily equivalent.

# Proof

$$\mathrm{tr}_A\{\rho_{AB}^{\alpha}\} \qquad \text{and} \qquad \left(1_A \otimes \rho_C^{\frac{\alpha-1}{2}}\right)\rho_{AC}\left(1_A \otimes \rho_C^{\frac{\alpha-1}{2}}\right)$$

- This is true since both of these operators are marginals — on $B$ and $AC$ — of the same tripartite rank-1 operator,

$$\left(1_{AB} \otimes \rho_C^{\frac{\alpha-1}{2}}\right)\rho_{ABC}\left(1_{AB} \otimes \rho_C^{\frac{\alpha-1}{2}}\right).$$

- To see this, note the first operator can be rewritten as

$$\begin{aligned}\mathrm{tr}_A\{\rho_{AB}^{\alpha}\} &= \mathrm{tr}_A\left\{\rho_{AB}^{\frac{\alpha-1}{2}}\,\rho_{AB}\,\rho_{AB}^{\frac{\alpha-1}{2}}\right\} \\ &= \mathrm{tr}_{AC}\left\{\left(\rho_{AB}^{\frac{\alpha-1}{2}} \otimes 1_C\right)\rho_{ABC}\left(\rho_{AB}^{\frac{\alpha-1}{2}} \otimes 1_C\right)\right\} \\ &= \mathrm{tr}_{AC}\left\{\left(1_{AB} \otimes \rho_C^{\frac{\alpha-1}{2}}\right)\rho_{ABC}\left(1_{AB} \otimes \rho_C^{\frac{\alpha-1}{2}}\right)\right\}.\end{aligned}$$

# Overview: Order and Duality

- Arrows indicate that one entropy is larger or equal to the other for all states $\rho_{AB}$ and all $\alpha$.

$$\begin{array}{ccc} H^{\uparrow}_{\alpha}(A|B)_{\rho} & \longleftarrow & \widetilde{H}^{\uparrow}_{\alpha}(A|B)_{\rho} \\ \downarrow & \swarrow & \downarrow \\ H^{\downarrow}_{\alpha}(A|B)_{\rho} & \longleftarrow & \widetilde{H}^{\downarrow}_{\alpha}(A|B)_{\rho} \end{array}$$

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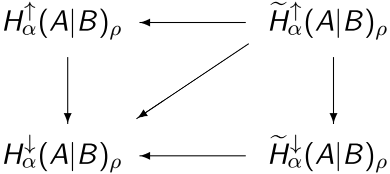


- For any pure $\rho_{ABC}$, the following duality relations hold:

$$H_\alpha^{\downarrow}(A|B)_\rho + H_\beta^{\downarrow}(A|C)_\rho = 0 \qquad \text{for} \quad \alpha, \beta \in [0, 2],\ \alpha + \beta = 2,$$

$$\widetilde{H}_\alpha^{\uparrow}(A|B)_\rho + \widetilde{H}_\beta^{\uparrow}(A|C)_\rho = 0 \qquad \text{for} \quad \alpha, \beta \in \left[\frac{1}{2}, \infty\right],\ \frac{1}{\alpha} + \frac{1}{\beta} = 2,$$

$$H_\alpha^{\uparrow}(A|B)_\rho + \widetilde{H}_\beta^{\downarrow}(A|C)_\rho = 0 \qquad \text{for} \quad \alpha, \beta \in [0, \infty],\ \alpha \cdot \beta = 1.$$

- The last equality is new. The first two are discussed in TCR'09 and MDSFT'13, respectively.

# Application: Entropy Inequalities

- The following inequalities hold for $\alpha \in \left[\frac{1}{2}, \infty\right]$:

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\begin{aligned}
H_\alpha^{\uparrow}(A|B)_\rho &\leq \widetilde{H}_\alpha^{\uparrow}(A|B)_\rho \leq H_{2-\frac{1}{\alpha}}^{\uparrow}(A|B)_\rho\,,\\
H_\alpha^{\downarrow}(A|B)_\rho &\leq H_\alpha^{\uparrow}(A|B)_\rho \leq H_{2-\frac{1}{\alpha}}^{\downarrow}(A|B)_\rho\,,\\
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\end{aligned}
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- Proof: Apply the duality relation for the different arrows:

$$
\begin{array}{ccc}
H_{\alpha}^{\uparrow}(A|B)_{\rho} & \longleftarrow & \widetilde{H}_{\alpha}^{\uparrow}(A|B)_{\rho} \\
\big\downarrow & & \big\downarrow \\
H_{\alpha}^{\downarrow}(A|B)_{\rho} & \longleftarrow & \widetilde{H}_{\alpha}^{\downarrow}(A|B)_{\rho}
\end{array}
$$

# Application: Entropy Inequalities

- This puts some known relations in a larger framework.
- For example, $\widetilde{H}^{\uparrow}_{\infty}(A|B)_\rho \leq \widetilde{H}^{\downarrow}_{2}(A|B)_\rho$ relates the conditional min-entropy to the conditional collision entropy.
- To understand this operationally we rewrite the conditional min-entropy as its dual semi-definite program (KRS'09),

$$\widetilde{H}^{\uparrow}_{\infty}(A|B)_\rho = \inf_{\Lambda_{B\to A'}} -\log\left(|A| \cdot F(\Phi_{AA'}, \Lambda_{B\to A'}[\rho_{AB}]\right),$$

  where $A'$ is a copy of $A$, the infimum is over all quantum channels $\Lambda_{B\to A'}$ and $\Phi_{AA'}$ is the maximally entangled state.
- The above inequality becomes apparent since the conditional collision entropy can be written as (BCS'13),

$$\widetilde{H}^{\downarrow}_{2}(A|B)_\rho = -\log\left(|A| \cdot F(\Phi_{AA'}, \Lambda^{\mathrm{pg}}_{B\to A'}[\rho_{AB}]\right),$$

  where $\Lambda^{\mathrm{pg}}_{B\to A'}$ denotes the pretty good recovery map of Barnum and Knill (BK'02).

# Application: Uncertainty Relations

- Duality relations + data processing (essentially) implies an entropic uncertainty relation (CCYZ'12).
- Let $\rho_{ABC}$ be a state and let $\{M_x\}_x$ and $\{N_y\}_y$ be two positive operator-valued measures.

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- Let $\rho_{ABC}$ be a state and let $\{M_x\}_x$ and $\{N_y\}_y$ be two positive operator-valued measures.
- We define *the overlap* $c := \max_{x,y} \left\| \sqrt{M_x} \sqrt{N_y} \right\|$.

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- We define *the overlap* $c := \max_{x,y} \left\| \sqrt{M_x} \sqrt{N_y} \right\|$.
- Consider the *post-measurement states*

$$\rho_{XB} := \bigoplus_x \mathrm{tr}_{AC} \left\{ M_x \rho_{ABC} \right\} \quad \text{and}$$

$$\rho_{YC} := \bigoplus_y \mathrm{tr}_{AB} \left\{ N_y \rho_{ABC} \right\} .$$

# Application: Uncertainty Relations

- The following generalizations of Maassen-Uffink hold:

$$
\begin{aligned}
H_\alpha^{\downarrow}(X|B)_\rho + H_\beta^{\downarrow}(Y|C)_\rho &\geq \log \frac{1}{c}, \\
&\text{for} \quad \alpha, \beta \in [0,2],\ \alpha + \beta = 2, \\
\widetilde{H}_\alpha^{\uparrow}(X|B)_\rho + \widetilde{H}_\beta^{\uparrow}(Y|C)_\rho &\geq \log \frac{1}{c}, \\
&\text{for} \quad \alpha, \beta \in \left[\frac{1}{2}, \infty\right],\ \frac{1}{\alpha} + \frac{1}{\beta} = 2, \\
H_\alpha^{\uparrow}(X|B)_\rho + \widetilde{H}_\beta^{\downarrow}(Y|C)_\rho &\geq \log \frac{1}{c}, \\
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- The last relation is new. The first two are discussed in CCYZ'12 and MDSFT'13, respectively.