Strong Converse Rates for Quantum Communication

Marco Tomamichel[1], Mark M. Wilde[2], Andreas Winter[3]

[1]School of Physics, The University of Sydney
[2]Center for Computation and Technology, Dept. of Physics & Astronomy, Louisiana State University
[3]ICREA & Física Teórica, Informació i Fenomens Quántics, Universitat Autònoma de Barcelona

THE UNIVERSITY OF SYDNEY

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Quantum Information Theory

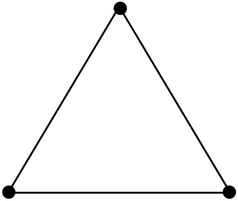
- Information theory is concerned with establishing fundamental limits for information processing tasks.
- The most precise modeling of physical information carriers (known today) uses the formalism of quantum mechanics.

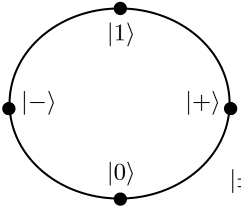
Source: Wikipedia

- Many fundamental questions remain unanswered, for example:
  1. No single-letter formula for point-to-point channel capacity.
  2. Strong converse for memoryless channels unknown.

Quantum Primer: States

classical world	**quantum world**
alphabet: $X = \{0, 1\}$	Hilbert space: $X = \mathbb{C}^2$ — "qubit" orthonormal basis: $\lvert 0\rangle, \lvert 1\rangle \in X$.
pmf: $P_X(x) = p_x$	quantum state: $\rho_X = \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix}$ generally: $\rho_X \in \mathcal{S}(X)$, linear operator on X with $\rho_X \geq 0$ and $\mathrm{tr}(\rho_X) = 1$.





$\lvert \pm\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle \pm \lvert 1\rangle)$

Quantum Primer: Correlation vs. Entanglement

- Relative entropy: $D(\rho \| \sigma) = \mathrm{tr}\left(\rho(\log \rho - \log \sigma)\right)$.
- Entropy: $H(A)_\rho = -D(\rho_A \| 1_A) = -\mathrm{tr}(\rho \log \rho)$.
- Cond. entropy: $H(A|B)_\rho = -D(\rho_{AB} \| 1_A \otimes \rho_B) = H(AB)_\rho - H(B)_\rho$.

classical world	**quantum world**
correlated pmf: $P_{XX'}(x,x') = \begin{cases} p_x & x = x' \\ 0 & \text{else} \end{cases}$	entangled quantum state: $\lvert\phi\rangle_{XX'} = \sqrt{p_0}\lvert 0\rangle \otimes \lvert 0\rangle + \sqrt{p_1}\lvert 1\rangle \otimes \lvert 1\rangle$ $\phi_{XX'} = \lvert\phi\rangle\langle\phi\rvert_{XX'} = \begin{pmatrix} p_0 & 0 & 0 & \sqrt{p_0 p_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{p_0 p_1} & 0 & 0 & p_1 \end{pmatrix}$
not extremal	extremal (pure, rank one)

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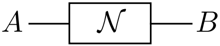
not extremal	extremal (pure, rank one)
entropy: $H(X)_P$ cond. entropy: $H(X\|X')_P = 0$	$H(X)_\phi = H(X)_P$ $H(X\|X')_\phi = -H(X)_P$

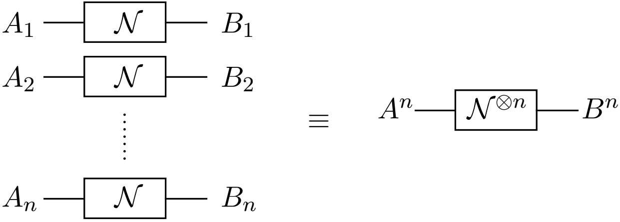
Quantum Coding: Channels

- **Quantum channel**: completely positive trace-preserving linear map $\mathcal{N} \equiv \mathcal{N}_{A\to B}$ from (states on) A to (states on) B.



  Assume A and B are finite-dimensional.

- The channel is memoryless:



Quantum Coding: Encoder and Decoder

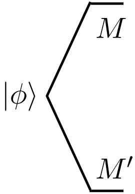
- **Entanglement transmission code** (for $\mathcal{N}^{\otimes n}$):

$$\mathcal{C}_n = \{d_n, \mathcal{E}_n, \mathcal{D}_n) \,.$$

1. code size d_n:
   - M, M', M'' of dimension d_n.
   - maximally entangled state

$$|\phi\rangle_{MM'} = \frac{1}{\sqrt{d_n}} \sum_{i=1}^{d_n} |i\rangle_M \otimes |i\rangle_{M'} \,.$$

2. encoder $\mathcal{E}_n$: quantum channel from M' to A^n.
3. decoder $\mathcal{D}_n$: quantum channel from B^n to M''.



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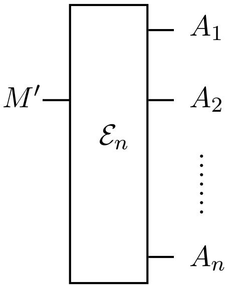

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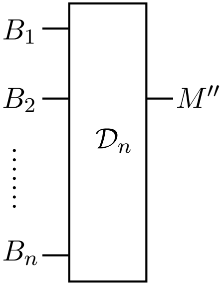
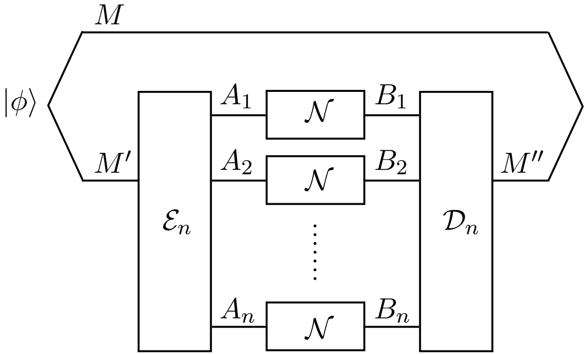


Quantum Coding: Entanglement Fidelity



- **Fidelity** with maximally entangled state:

$$F(\mathcal{C}_n, \mathcal{N}^{\otimes n}) = \operatorname{tr}\left((\mathcal{D}_n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_n)(\phi_{MM'})\phi_{MM''}\right).$$

- Corresponds to $\Pr[M = M'']$ classically.

Quantum Capacity

- A triple (R, n, ε) is achievable on $\mathcal{N}$ if $\exists\, \mathcal{C}_n$ with

$$\frac{1}{n} \log d_n \geq R, \quad \text{and} \quad F(\mathcal{C}_n \mathcal{N}^{\otimes n}) \geq 1 - \varepsilon\,.$$

- Boundary of (non-asymptotic) achievable region:

$$\hat{R}(n; \varepsilon, \mathcal{N}) := \max \big\{ R : (R, n, \varepsilon) \text{ is achievable on } \mathcal{N} \big\}.$$

- The *quantum capacity*, $Q(\mathcal{N})$, is the rate at which qubits can be transmitted with fidelity approaching one asymptotically.

$$Q_\varepsilon(\mathcal{N}) := \lim_{n \to \infty} \hat{R}(n; \varepsilon, \mathcal{N}), \qquad \varepsilon \in (0,1)$$
$$Q(\mathcal{N}) := \lim_{\varepsilon \to 0} Q_\varepsilon(\mathcal{N})\,.$$

Quantum Capacity Theorem

- Barnum, Nielsen and Schumacher (1996-2000) as well as Lloyd, Shor and Devetak (1997-2005) established

$$Q(\mathcal{N}) = \lim_{\ell \to \infty} \frac{1}{\ell} I_{\mathrm{c}}(\mathcal{N}^{\otimes \ell}), \qquad \text{where}$$

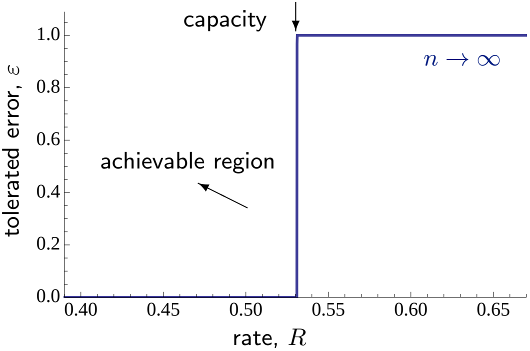
$$I_{\mathrm{c}}(\mathcal{N}) = \max_{\rho_A} \{-H(A|B)_\omega\}, \quad \omega_{AB} = \mathcal{N}_{A' \to B}(\psi^{\rho}_{A'A})$$

- This result is unsatisfactory for many reasons:
  1. It is not a single-letter formula.
  2. The limit $\ell \to \infty$ is necessary in general (Cubitt *et al.*'14).
  3. It cannot be calculated except for e.g. degradable channels which satisfy $I_{\mathrm{c}}(\mathcal{N}^{\otimes n}) = n I_{\mathrm{c}}(\mathcal{N})$.
  4. We do not know anything about $Q_\varepsilon(\mathcal{N})$.

Strong Converse Property

- A quantum channel satisfies the *strong converse property* iff

$$Q_\varepsilon(\mathcal{N}) = Q(\mathcal{N}) \quad \text{for all} \quad \varepsilon \in (0,1)\,.$$



- A *strong converse rate* is a uniform upper bound on Q_ε.

State of the Art

- **Until this work, the strong converse property could only be established for some channels with trivial capacity.**
- Morgan and Winter showed that *degradable quantum channels* satisfy a "pretty strong" converse:

$$Q_\varepsilon(\mathcal{N}) = Q(\mathcal{N}) \quad \text{for all } \varepsilon \in \left(0, \frac{1}{2}\right)$$

(Extending their proof to all $\varepsilon \in (0, 1)$ appears difficult.)

- Strong converse rates are known, for example the *entanglement-assisted capacity* established via channel simulation by Bennett *et al.* and Berta *et al.*
- However, they are not tight except for trivial channels.

A lot of (fundamental) work still needs to be done!

Result 1: Rains Entropy is Strong Converse Rate

- The *Rains relative entropy* of the channel is defined as

$$R(\mathcal{N}) := \max_{\rho_A} \min_{\sigma_{AB} \in \mathrm{Rains}(A:B)} D\left(\mathcal{N}_{A' \to B}(\psi^{\rho}_{A'A}) \,\middle\|\, \sigma_{AB}\right).$$

- A state $\sigma_{AB} \in \mathrm{Rains}(A:B)$ (cf. Rains'99) satisfies

$$\mathrm{tr}\left(\phi_{AB}\sigma_{AB}\right) \leq \frac{1}{d} \quad \forall \text{ maximally entangled } \phi_{AB}.$$

Result

For any channel $\mathcal{N}$, communication at a rate exceeding $R(\mathcal{N})$ implies (exponentially) vanishing fidelity.

- Key Idea: Consider correlations σ_{AB} that are useless for quantum communication. Classically:

$$C(W) = \max_{P_X} \min_{Q_X, Q_Y} D\left(P_X \times W_{Y|X} \middle\| Q_X \times Q_Y\right).$$

Step 1: Arimoto-Type (One-Shot) Converse Bound

- Consider $\mathcal{C} = \{d, \mathcal{E}, \mathcal{D}\}$ for $\mathcal{N}$ with $F(\mathcal{C}, \mathcal{N}) \geq 1 - \varepsilon$.
- Test if a state is $\phi_{MM''}$, or not:

$$\mathcal{T}(\cdot) = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \quad p = \mathrm{tr}(\phi_{MM''} \cdot \,).$$

- Let $\rho_{AM} = \mathcal{E}(\phi_{MM'})$. Due to data-processing, we have

$$\begin{aligned}\widetilde{D}_\alpha(\mathcal{N}(\rho_{AM}) \| \sigma_{BM}) &\geq D_\alpha\big(\mathcal{T} \circ \mathcal{D} \circ \mathcal{N}(\rho_{AM}) \| \mathcal{T} \circ \mathcal{D}(\sigma_{RB})\big) \\ &\geq \log d + \frac{\alpha}{\alpha - 1} \log(1 - \varepsilon),\end{aligned}$$

  for Rényi divergence with $\alpha > 1$
- Sandwiched Rényi divergence (Lennert *et al.*, Wilde *et al.*'13):

$$\widetilde{D}_\alpha(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \mathrm{tr} \left(\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right).$$

Step 2: Asymptotics

- Minimizing $\sigma_{AB} \in \mathrm{Rains}(A:B)$ and optimizing over codes:

Lemma

We have the following one-shot converse:

$$\hat{R}(1;\varepsilon,\mathcal{N}) \leq \max_{\rho_A} \min_{\sigma_{AB}} \widetilde{D}_\alpha(\mathcal{N}_{A'\to B}(\psi^\rho_{AA'})\|\sigma_{AB}) + \frac{\alpha \log \frac{1}{1-\varepsilon}}{\alpha - 1}$$

- This yields an upper bound on the ε-capacity:

$$Q_\varepsilon(\mathcal{N}) \leq \limsup_{n\to\infty} \frac{1}{n} \underbrace{\max_{\rho_{A^n}} \min_{\sigma_{A^nB^n}} \widetilde{D}_\alpha\Big(\mathcal{N}^{\otimes n}(\psi^\rho_{A^nA'^n})\Big\|\sigma_{A^nB^n}\Big)}_{\widetilde{R}_\alpha(\mathcal{N}^{\otimes n})} .$$

- It remains to show that $\widetilde{R}_\alpha(\mathcal{N})$ satisfies an asymptotic sub-additivity property, i.e. $\widetilde{R}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{R}_\alpha(\mathcal{N}) + o(n)$.

Step 3: Covariant Channels and Permutations

- Covariance group of the channel $\mathcal{N}$: Group G with unitary representations U_A and V_B such that

$$\mathcal{N}_{A\to B}(U_A(g)(\,\cdot\,)U_A^\dagger(g)) = V_B(g)\mathcal{N}_{A\to B}(\,\cdot\,)V_B^\dagger(g) \quad \forall g \in G$$

- We show that:

$$\widetilde{R}_\alpha(\mathcal{N}) = \max_{\bar{\rho}_A} \min_{\sigma_{AB}} \widetilde{D}_\alpha\big(\mathcal{N}_{A'\to B}(\psi^{\bar{\rho}}_{AA'})\,\big\|\,\sigma_{AB}\big)$$

where $\bar{\rho}_A = U_A(g)\bar{\rho}_A U_A^\dagger(g)$, i.e. $\bar{\rho}_A$ is invariant under G.

- Covariance group of $\mathcal{N}^{\otimes n}$ always contains permutations S_n.
- Thus, we can restrict the optimization in $\widetilde{R}_\alpha(\mathcal{N}^{\otimes n})$ to permutation invariant states $\bar{\rho}_{A^n}$.

Step 4: Asymptotic Sub-Additivity

- Employ the fact that $\psi^{\bar{\rho}}_{A^n A'^n}$ is in the symmetric subspace:

$$\psi^{\bar{\rho}}_{AA'} \leq P^{\mathrm{symm}}_{A^n R^n} \leq n^{|A|^2} \int \mathrm{d}\mu(\theta)\, \theta^{\otimes n}_{AR} \,.$$

- The quantum way to restrict to product states in the converse.
- This allows us to show (skipping a few technical steps) that

$$\widetilde{R}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{R}_\alpha(\mathcal{N}) + O(\log(n)).$$

- Hence, $Q_\varepsilon(\mathcal{N}) \leq \widetilde{R}_\alpha(\mathcal{N})$ for all $\alpha > 1$.
- And, thus, by continuity as $\alpha \to 1$, we find $Q_\varepsilon(\mathcal{N}) \leq R(\mathcal{N})$.
- A more detailed analysis reveals that the fidelity converges exponentially fast to 0 for any $d > R(\mathcal{N})$.

Result 2: Dephasing Channels Satisfy Strong Converse

- For all quantum channels we thus have

$$I_{\mathrm{c}}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q_{\varepsilon}(\mathcal{N}) \leq R(\mathcal{N})$$

for all $\varepsilon \in (0,1)$.

Result

For generalized dephasing channels $\mathcal{Z}$, we have $I_c(\mathcal{Z}) = R(\mathcal{Z})$.

- The inequalities collapse and $Q_{\varepsilon}(\mathcal{Z}) = Q(\mathcal{Z})$.
- Includes qubit dephasing channel:

$$\mathcal{Z}_{\lambda} : \rho \mapsto (1-\lambda)\rho + \lambda Z\,\rho\,Z\,.$$

$$\begin{pmatrix} a & c \\ c^{\dagger} & b \end{pmatrix} \mapsto \begin{pmatrix} a & (1-2\lambda)c \\ (1-2\lambda)c^{\dagger} & b \end{pmatrix}$$

Summary and Outlook

- Our strong converse also holds with one-way classical communication assistance. However, it is unclear if it also holds for two-way assistance.
- For more details on this result, see:

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arXiv: 1406.2946

- The techniques also allow to establish a one-shot "meta-converse" in terms of binary quantum hypothesis testing (see arXiv: 1504.04617).