Entropic Uncertainty Relations and their Applications in Quantum Cryptography

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Take Home Message

Entropic uncertainty relations allow us to formalize physical intuition in information theoretic terms.

$+$

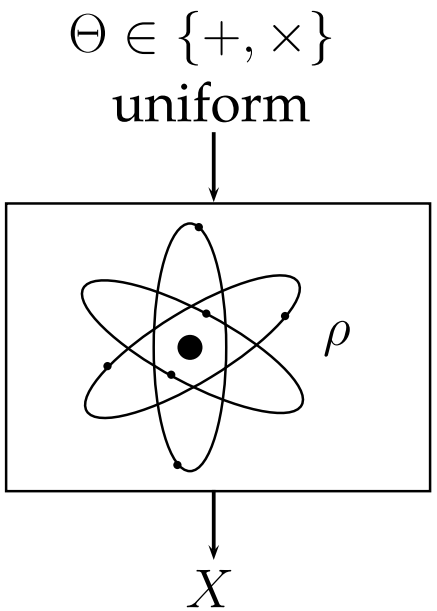
The min-entropy is the preferred measure of uncertainty for cryptographic applications.

$=$

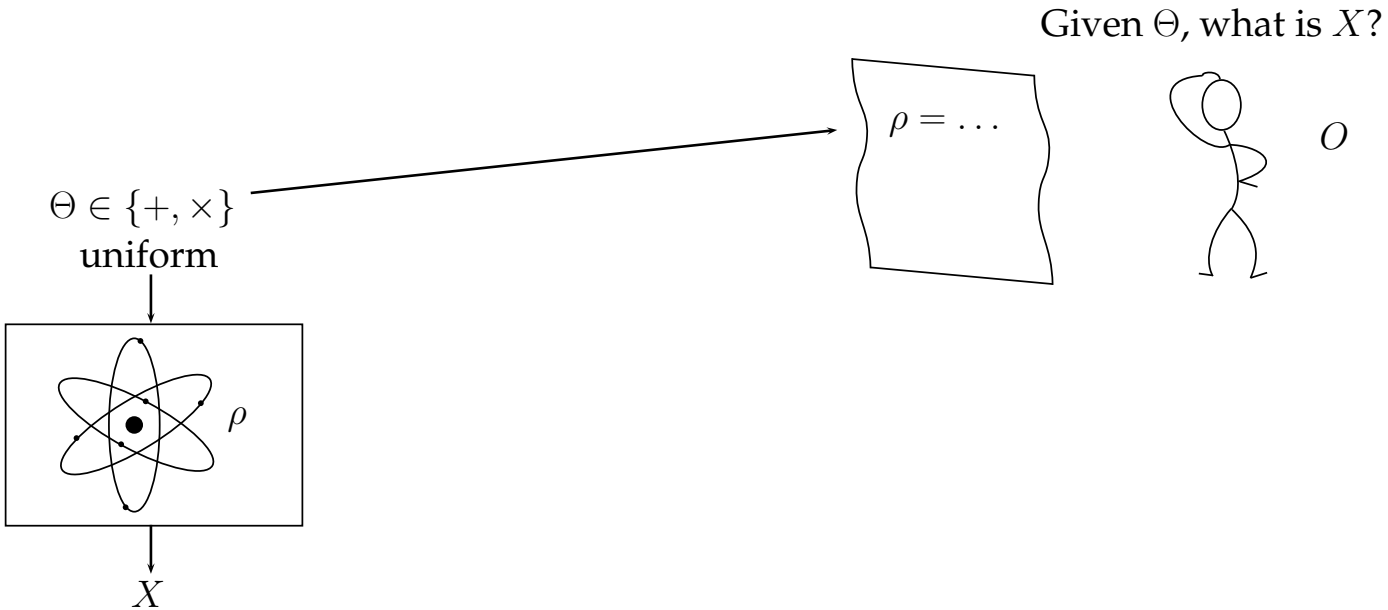
Uncertainty relations in terms of the min-entropy consequently have a wide range of applications in quantum cryptography.

From Quantum Systems to Random Variables

- Simplest example: Measure photon polarization in rectilinear '$+$' or diagonal '$\times$' basis.
- More generally, we consider two arbitrary measurements on an arbitrary quantum system in state ρ.



Uncertainty Relation with Classical Observer



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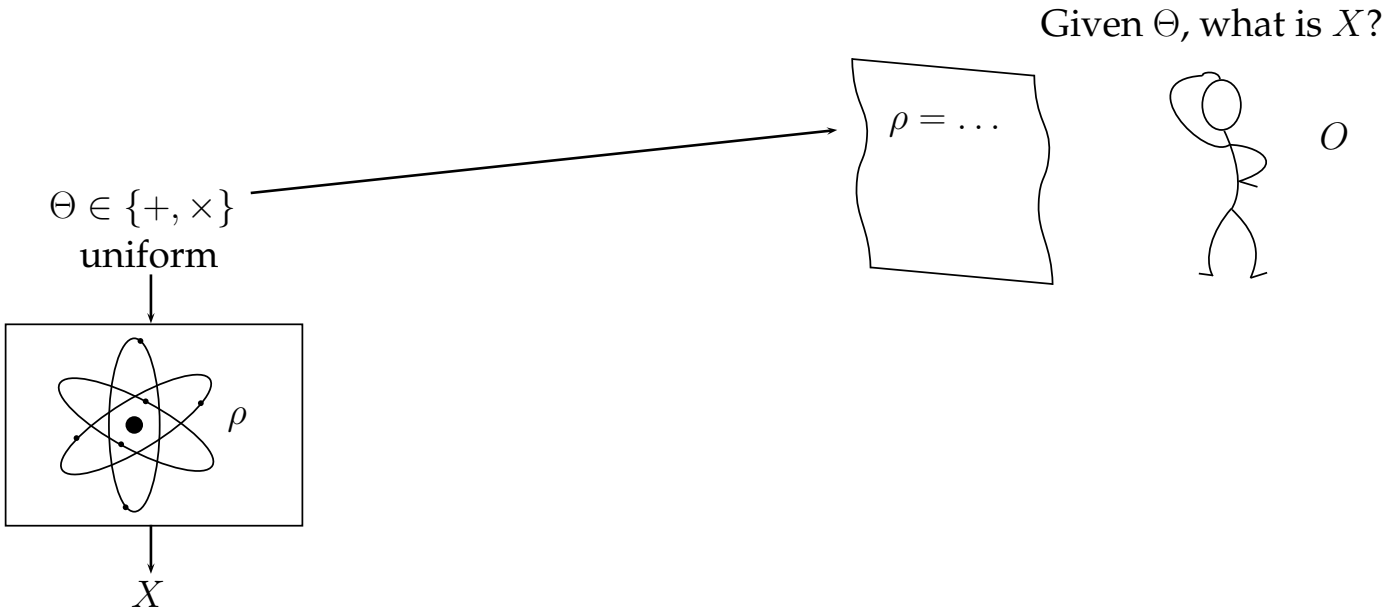
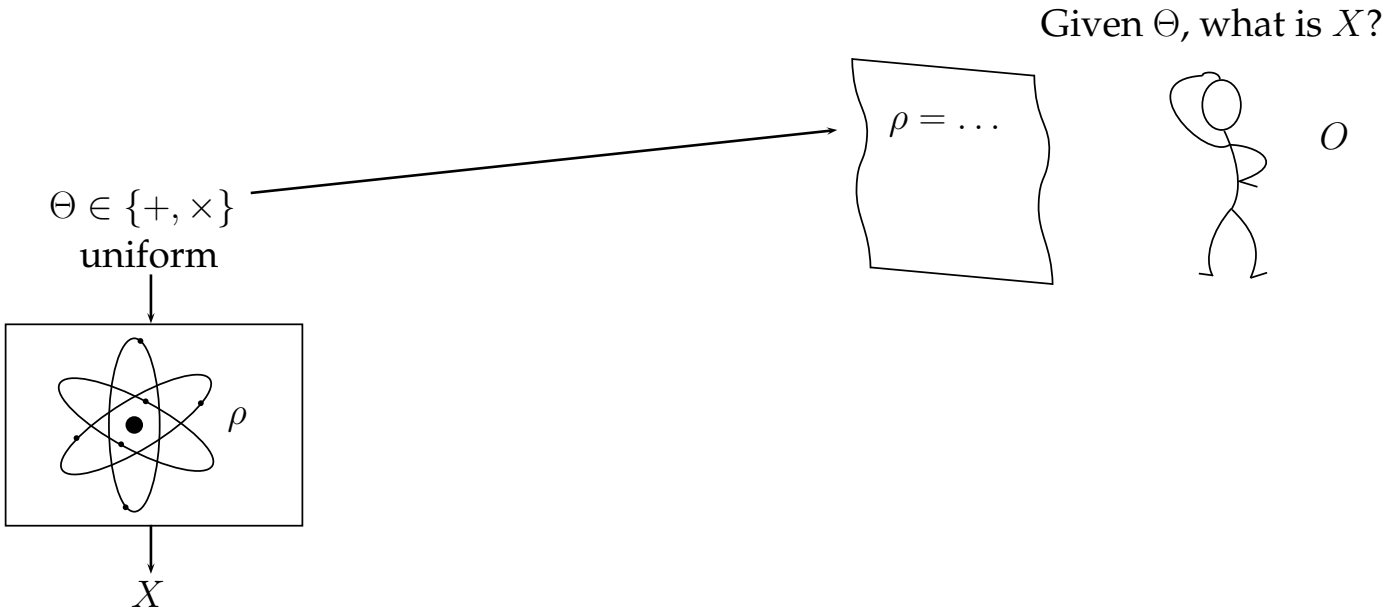


- Heisenberg's uncertainty principle [Hei27]:
  It is impossible to predict the outcome of both measurements.

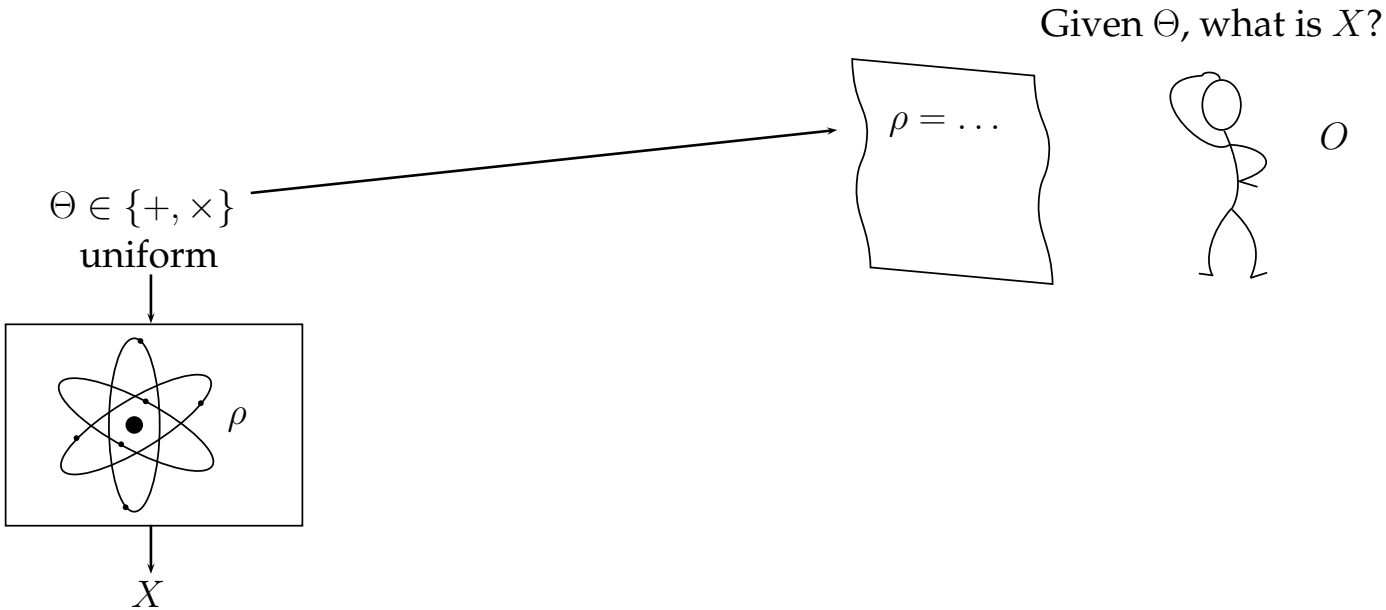
Uncertainty Relation with Classical Observer



- Heisenberg's uncertainty principle [Hei27]:
  It is impossible to predict the outcome of both measurements.
- Deutsch [Deu83] proposed to characterize this uncertainty in terms of the Shannon entropy [Sha48].

$$H(X) := \sum_x P_X(x) \log \frac{1}{P_X(x)}, \quad H(X|Y) := \sum_y P_Y(y)\, H(X|Y=y).$$

Uncertainty Relation with Classical Observer



- Simplest example: measure polarization for photon prepared in the rectilinear '+' basis.

$$H(X|O,\Theta) = \frac{1}{2}H(X|O,\Theta='+') + \frac{1}{2}H(X|O,\Theta='\times').$$

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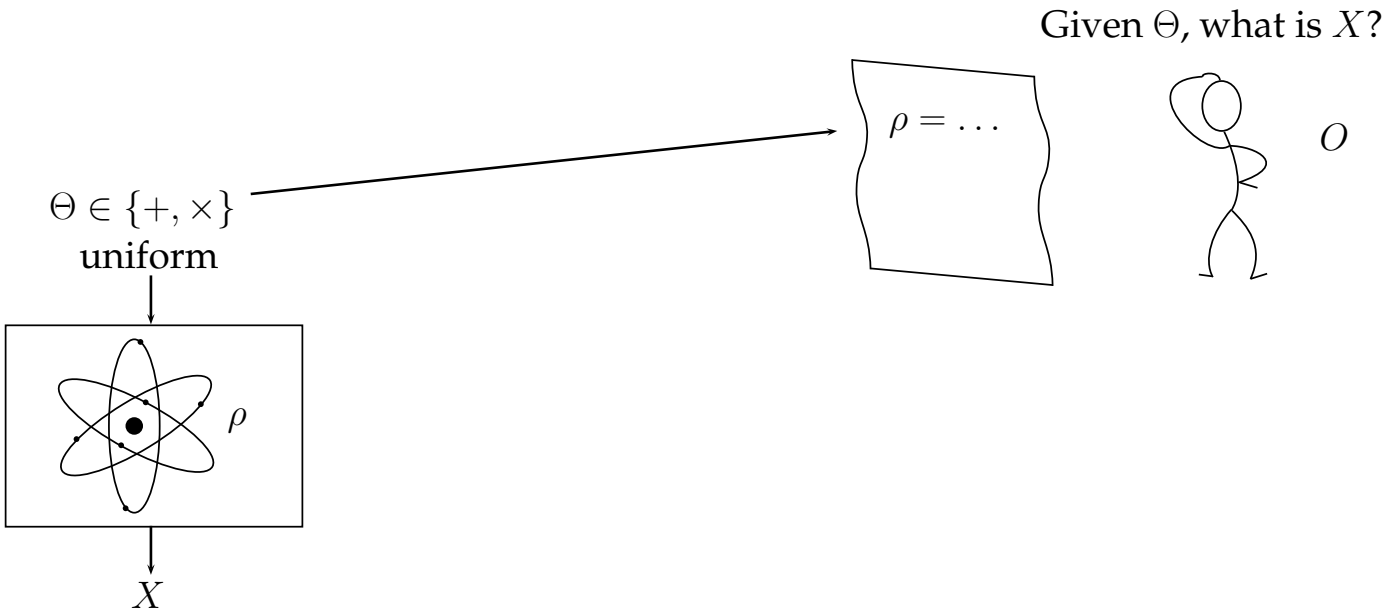

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$$H(X|O,\Theta) = \frac{1}{2}\underbrace{H(X|O,\Theta='+')}_{=0} + \frac{1}{2}\underbrace{H(X|O,\Theta='\times')}_{=1}.$$

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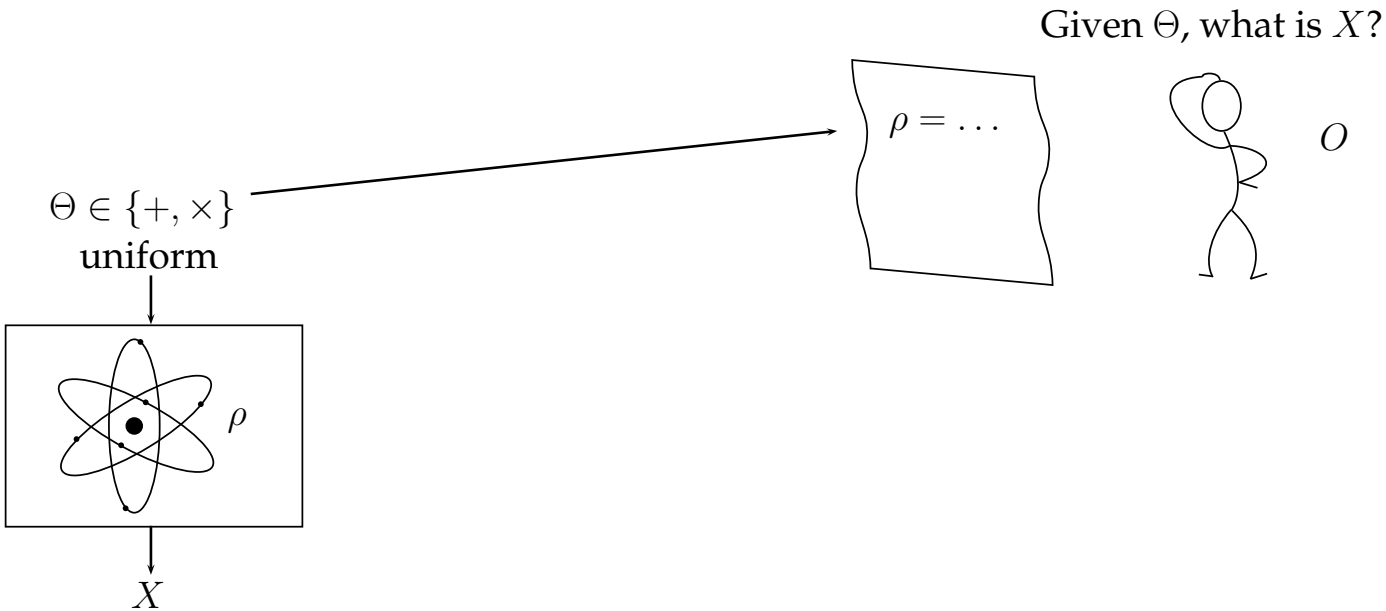

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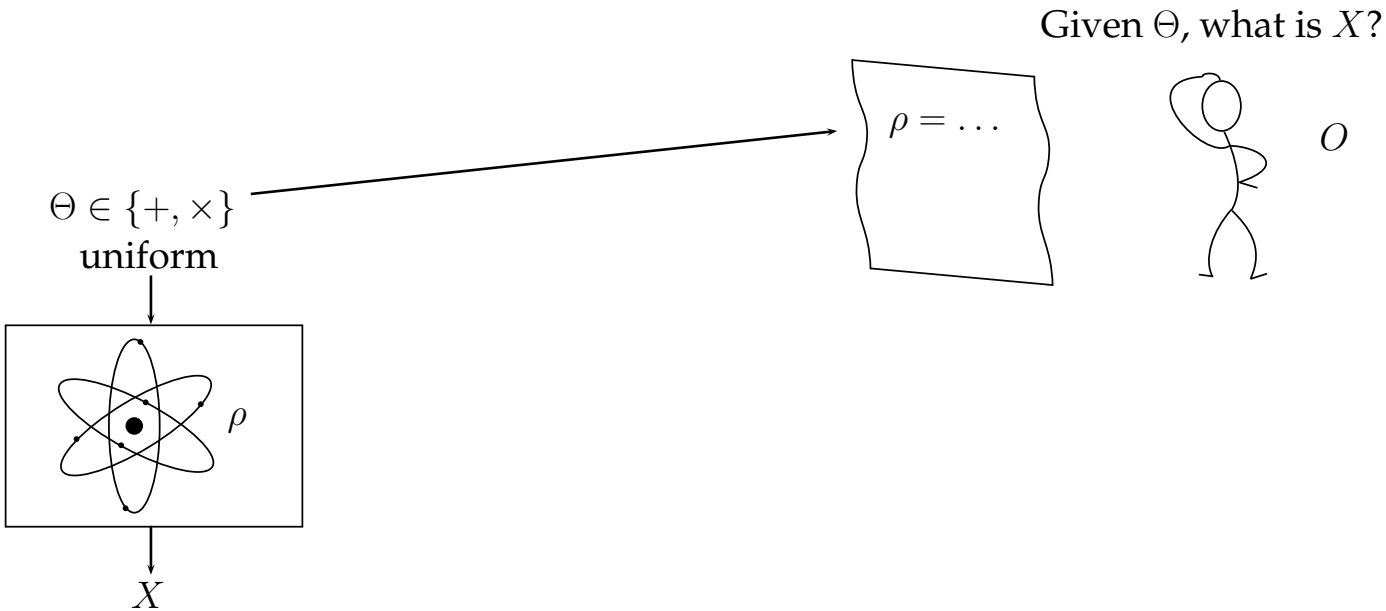

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$$H(X|O,\Theta) = \frac{1}{2}H(X|O,\Theta='+') + \frac{1}{2}H(X|O,\Theta='\times') = \frac{1}{2}.$$

- In fact, the bound $H(X|O,\Theta) \geq \frac{1}{2}$ holds independently of how the state ρ is prepared!

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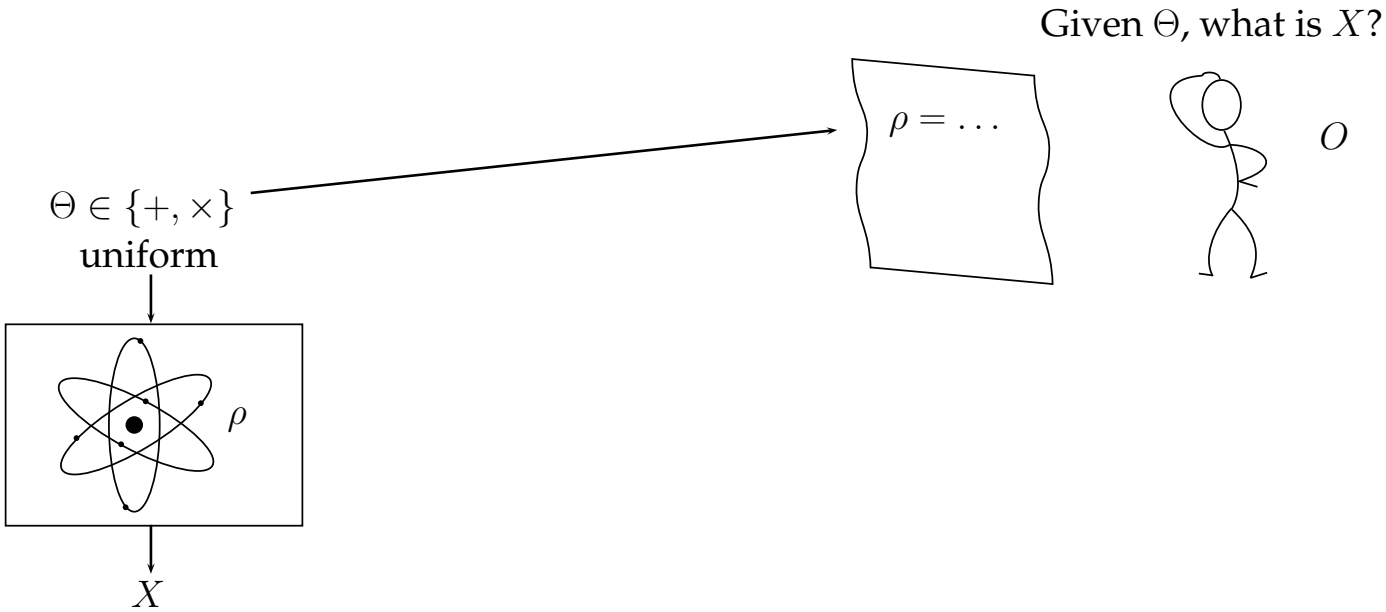


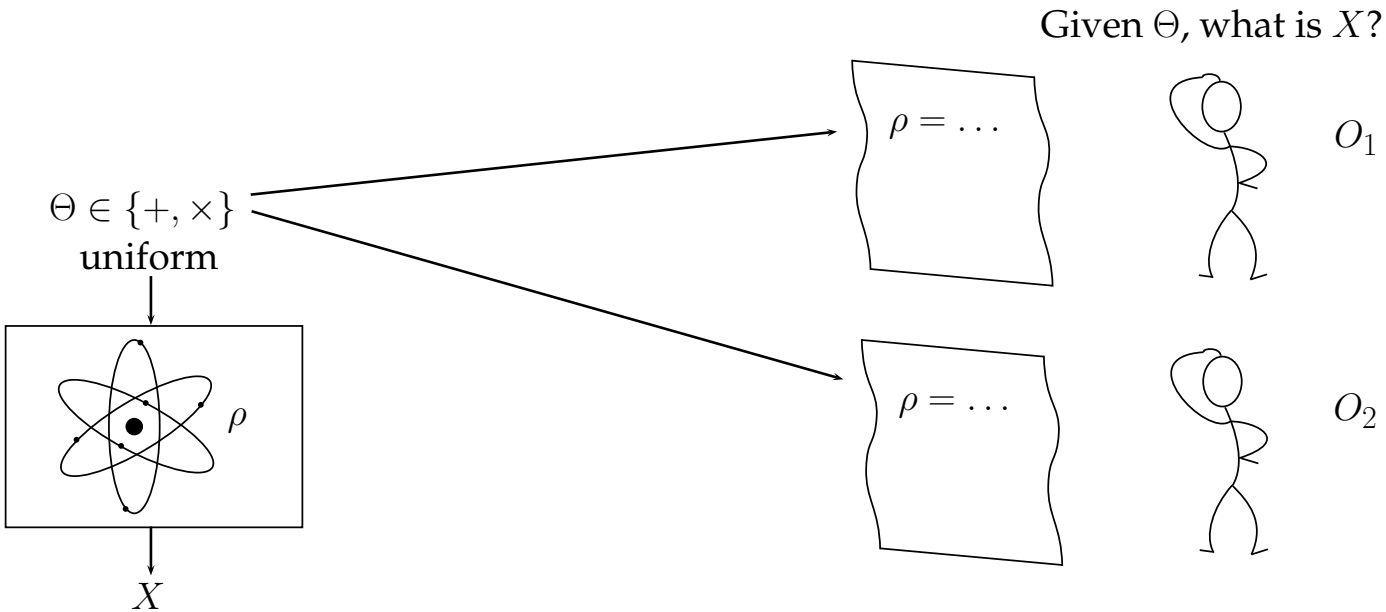
More generally, for arbitrary projective measurements:

Maassen & Uffink [MU88]

$$H(X|O,\Theta) \geq \frac{1}{2}\log\frac{1}{c}, \quad c = \max_{x,y} \left|\langle \phi_x^+ | \phi_y^\times \rangle\right|^2,$$

where c is called *overlap* and $\left\{|\phi_x^+\rangle\right\}_x$ and $\left\{|\phi_y^\times\rangle\right\}_y$ are the eigenvectors of the respective measurements.

Uncertainty Relation with Two Classical Observers

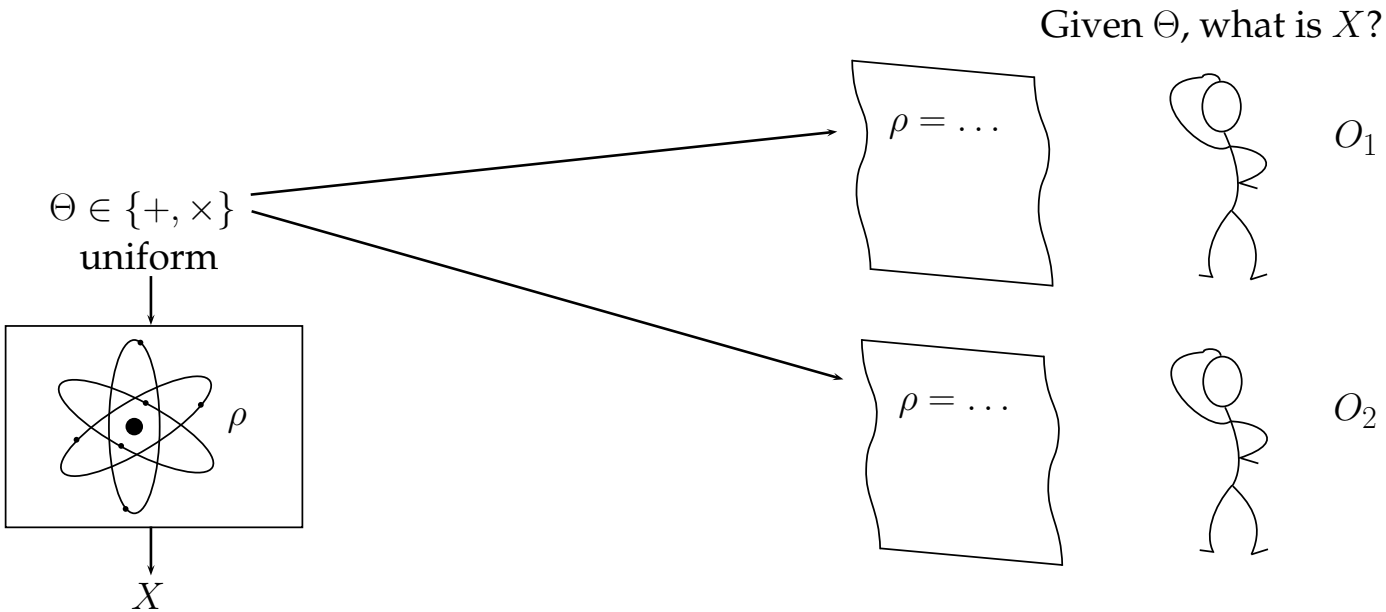


The previous relations trivially extends to two observers:

Maassen & Uffink [MU88]

$$H(X|O_1, \Theta) + H(X|O_2, \Theta) \geq \log_2 \frac{1}{c} .$$

Uncertainty Relation with Two Classical Observers



More generally, the relation also holds for Rényi entropies [Rén61]:

Maassen & Uffink [MU88]

$$H_\alpha(X|O_1,\Theta) + H_\beta(X|O_2,\Theta) \geq \log \frac{1}{c}, \quad \text{where} \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2.$$

Classical Rényi Entropy Intermezzo

- Rényi [Rén61] defined a family of divergences for $\alpha \in [0, \infty]$ as

$$D_\alpha(P\|Q) := \frac{1}{\alpha - 1} \log \sum_z P(z)^\alpha Q(z)^{1-\alpha},$$

taking limits where necessary.

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  taking limits where necessary.
- The Rényi entropy is related to the divergence,

$$H_\alpha(X) := \frac{1}{1-\alpha} \log \sum_x P_X(x)^\alpha = -D_\alpha(P_X\|1_X)\,.$$

- The proper extension to conditional Rényi entropy is disputed. Here we employ a definition due to Arimoto [Ari73]:

$$H_\alpha(X|Y) := \frac{\alpha}{1-\alpha} \log \left(\sum_y P_Y(y) \exp\left(\frac{1-\alpha}{\alpha} H_\alpha(X|Y=y) \right) \right)$$

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$$= \frac{\alpha}{1-\alpha} \log \left(\sum_y P_Y(y) \left(\sum_x P_{X|Y}(x)^\alpha \right)^{\frac{1}{\alpha}} \right).$$

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- This extension is closely related to the Rényi divergence,

$$H_\alpha(X|Y) = \sup_{Q_Y} -D_\alpha(P_{XY}\|1_X \times Q_Y).$$

Classical Rényi Entropy Intermezzo

- The function $\alpha \to H_\alpha(X|Y)$ is monotonically decreasing.
- The Shannon entropy is $\lim_{\alpha\to 1} H_\alpha(X|Y) = H(X|Y)$.
- It satisfies a data-processing inequality. For any channel $Y \to Z$, we have

$$H_\alpha(X|Y) \leq H_\alpha(X|Z).$$

  Operationally, the uncertainty of X increases when we manipulate the side information Y.
- The quantity has found operational significance. In particular,

$$\lim_{\alpha\to\infty} H_\alpha(X|Y) = H_{\min}(X|Y) = -\log p_{\text{guess}}(X|Y),$$

  where $p_{\text{guess}}(X|Y) = \sum_y P_Y(y) \max_x P_{X|Y}(x)$ is the (average) probability of correctly guessing X given Y.

Min-Entropy and "Leftover Hashing"

- The omnipresence of the min-entropy in cryptography is due to the "Leftover Hashing" Lemma.

Bennett *et al.* [BBR88] & Impagliazzo *et al.* [ILL89, IZ89]

Let $X \in \mathcal{X}$ be such that $H_{\min}(X|E) \geq k$, and $\ell \in \mathbb{N}$. Then, there exists a family of Hash function $\{f_s\}_s$ from $\mathcal{X}$ to $\{0,1\}^\ell$ such that $f_S(X)$ is δ-close to uniform and independent of E and S, where

$$\delta \sim \sqrt{2^{\ell-k}}$$

and S is chosen uniformly at random.

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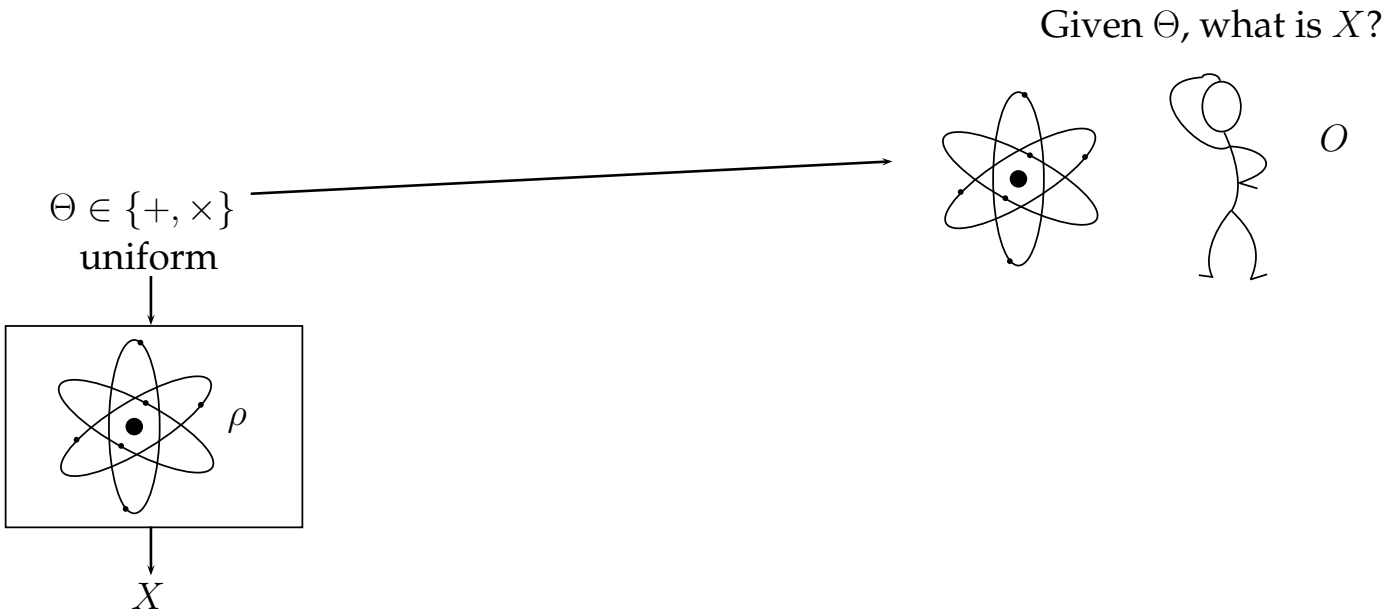
and S is chosen uniformly at random.

- This is "tight" in the sense that one cannot expect to extract more than $H_{\min}^{\varepsilon}(X|E)$ bits of uniform and independent randomness.

Uncertainty vs. Entanglement



- Consider a joint quantum state ρ_{AO}.

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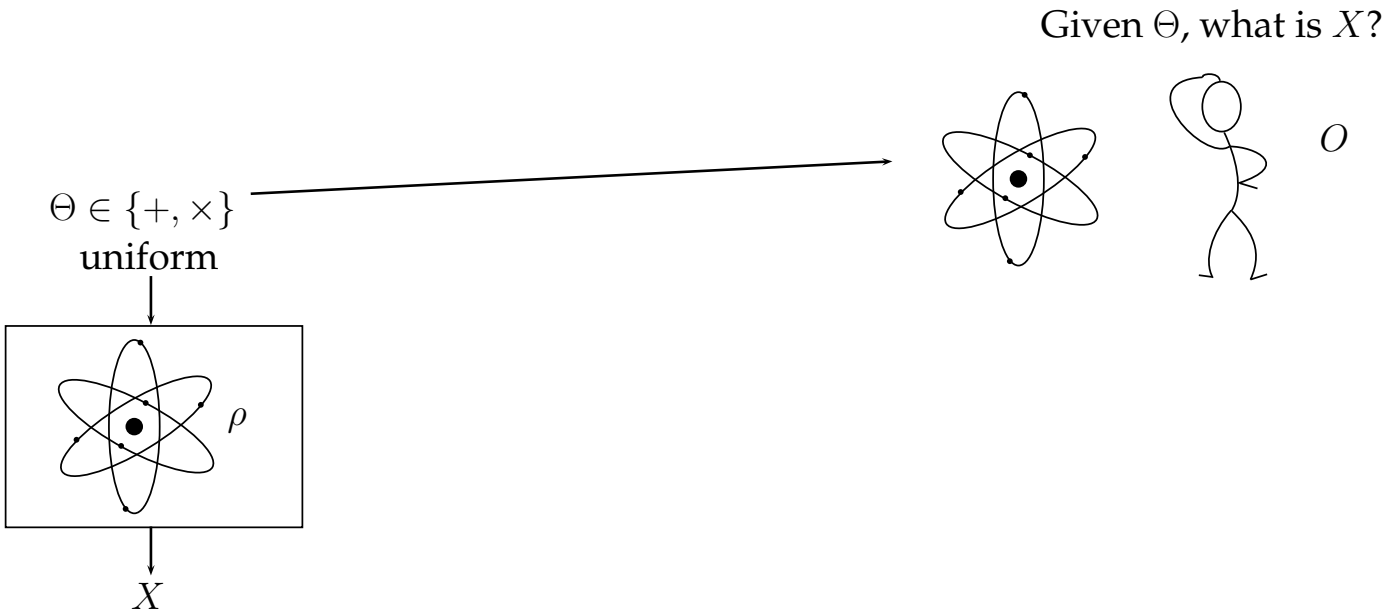

- Consider a joint quantum state ρ_{AO}.
- If $\rho_{AO} = |\psi\rangle\langle\psi|$ is maximally entangled then $H(X|O,\Theta) = 0$.
  - Why? The observer may choose a measurement on O — depending on Θ — to get perfect correlation with X.

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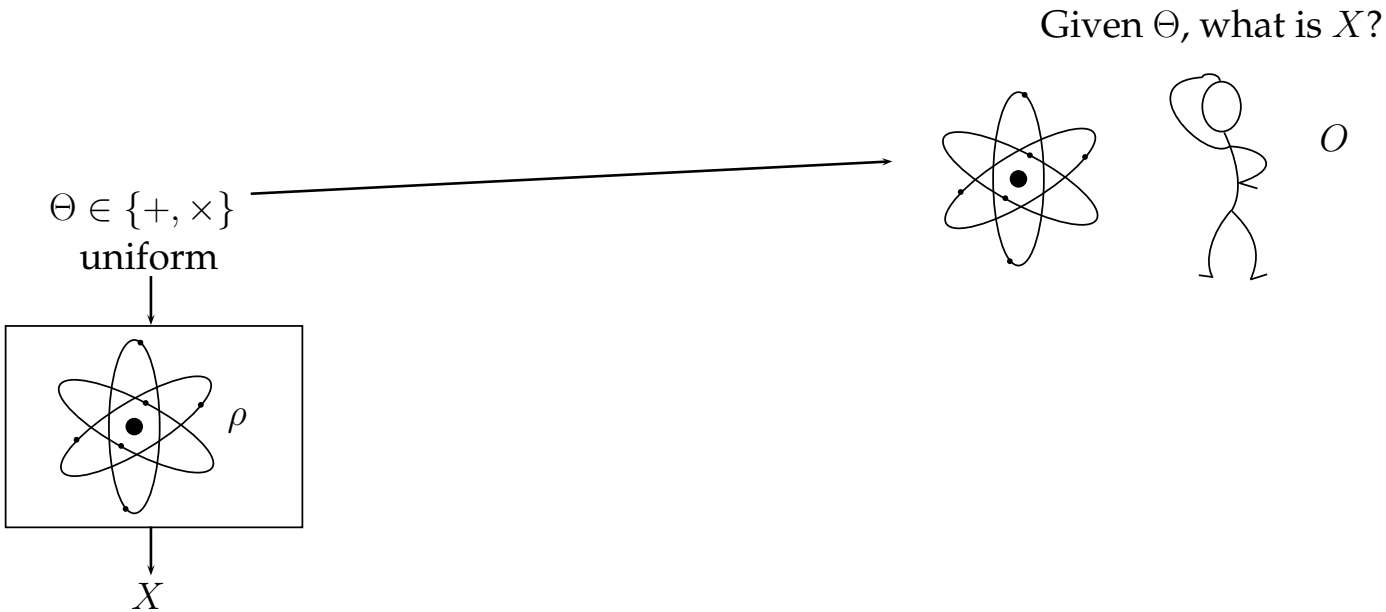

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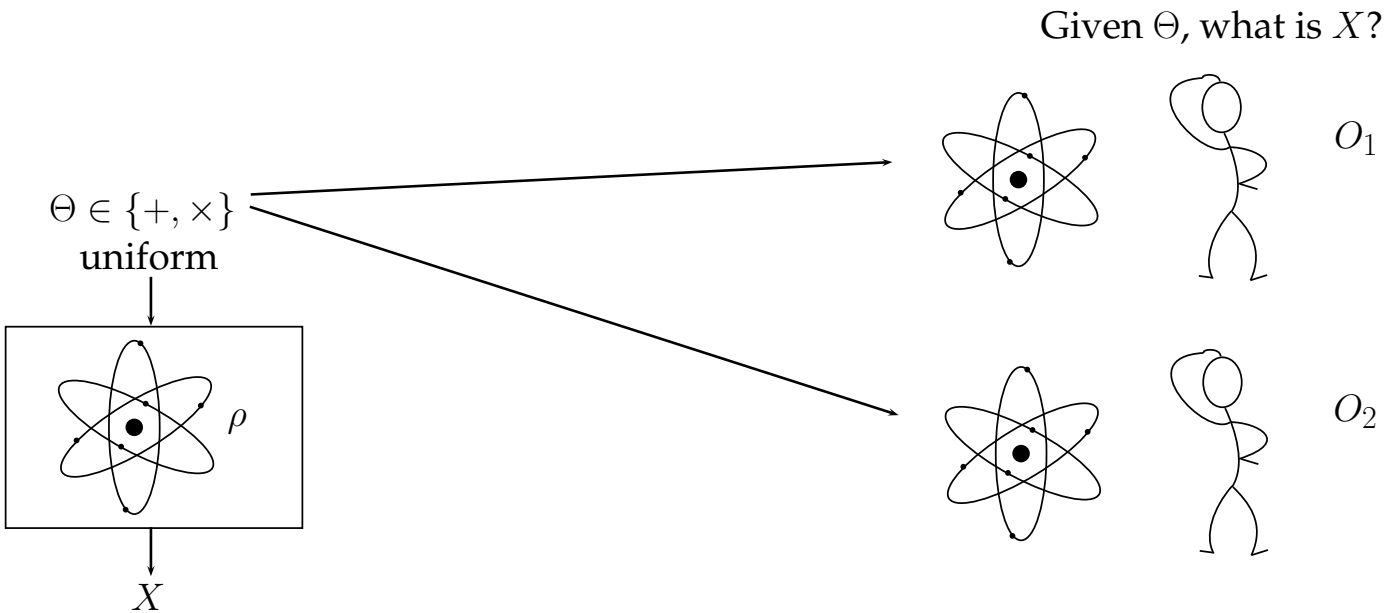
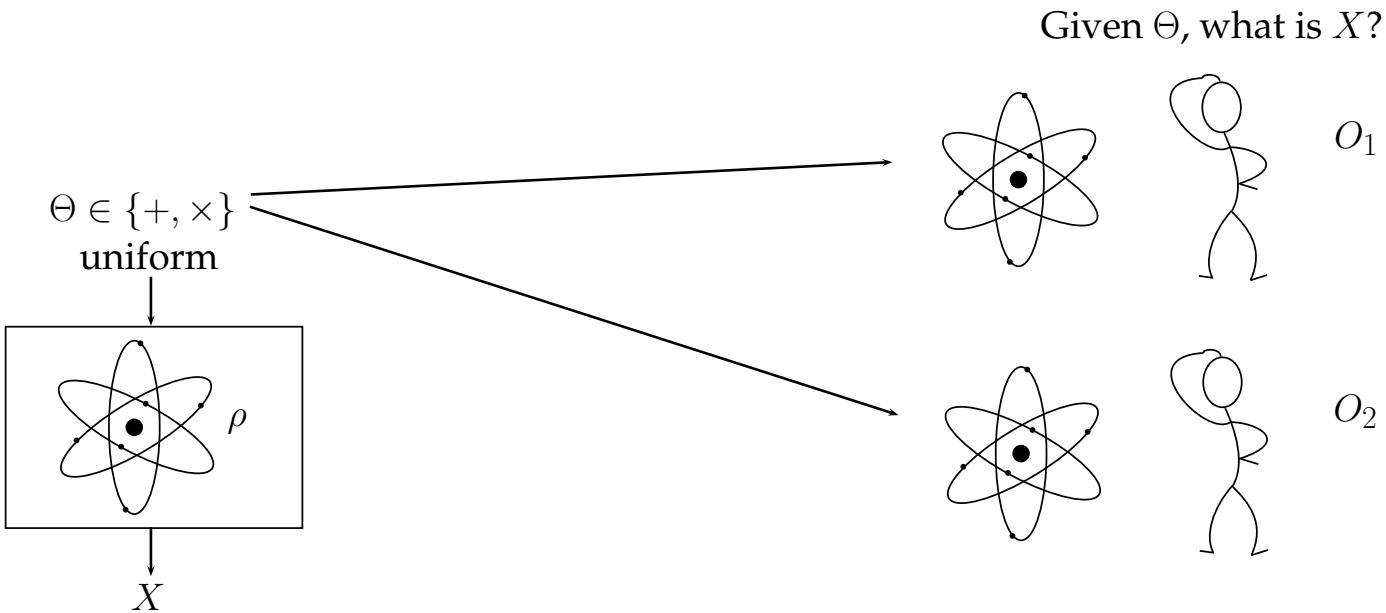
  - Why? The observer may choose a measurement on O — depending on Θ — to get perfect correlation with X.
- No uncertainty relation here!

Uncertainty and Monogamy of Entanglement



Can monogamy of entanglement help?

Uncertainty and Monogamy of Entanglement



Can monogamy of entanglement help?

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The post-measurement states satisfy

$$H(X|O_1, \Theta) + H(X|O_2, \Theta) \geq \log_2 \frac{1}{c} .$$

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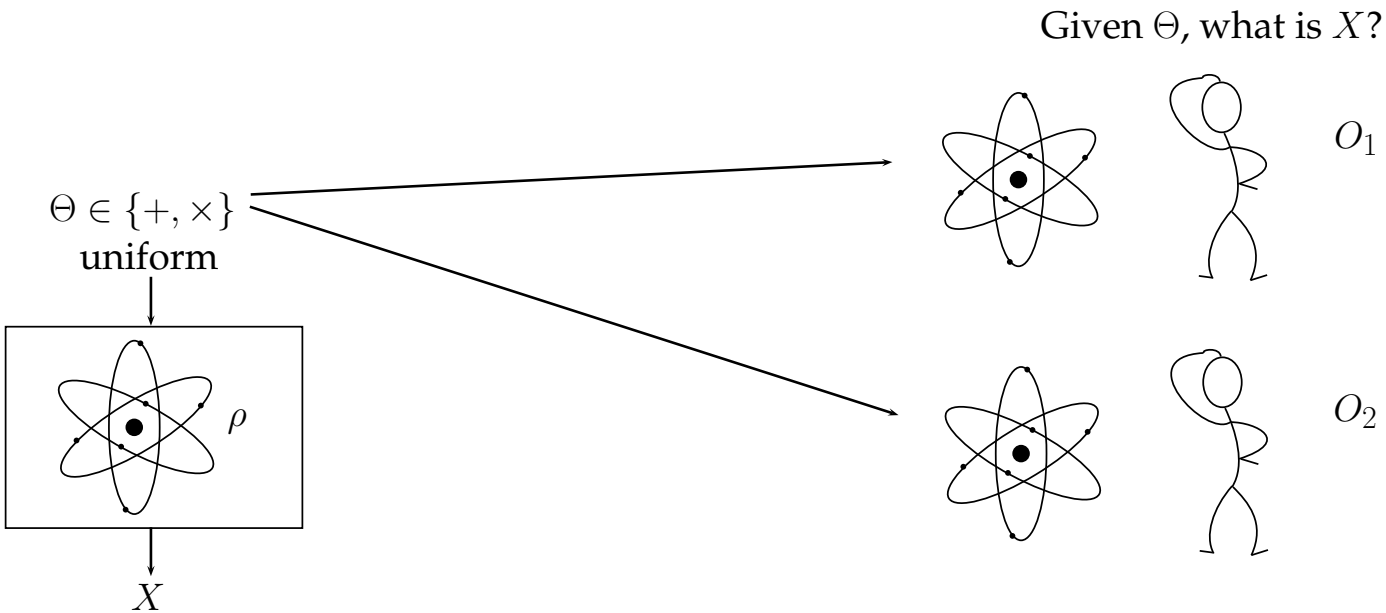

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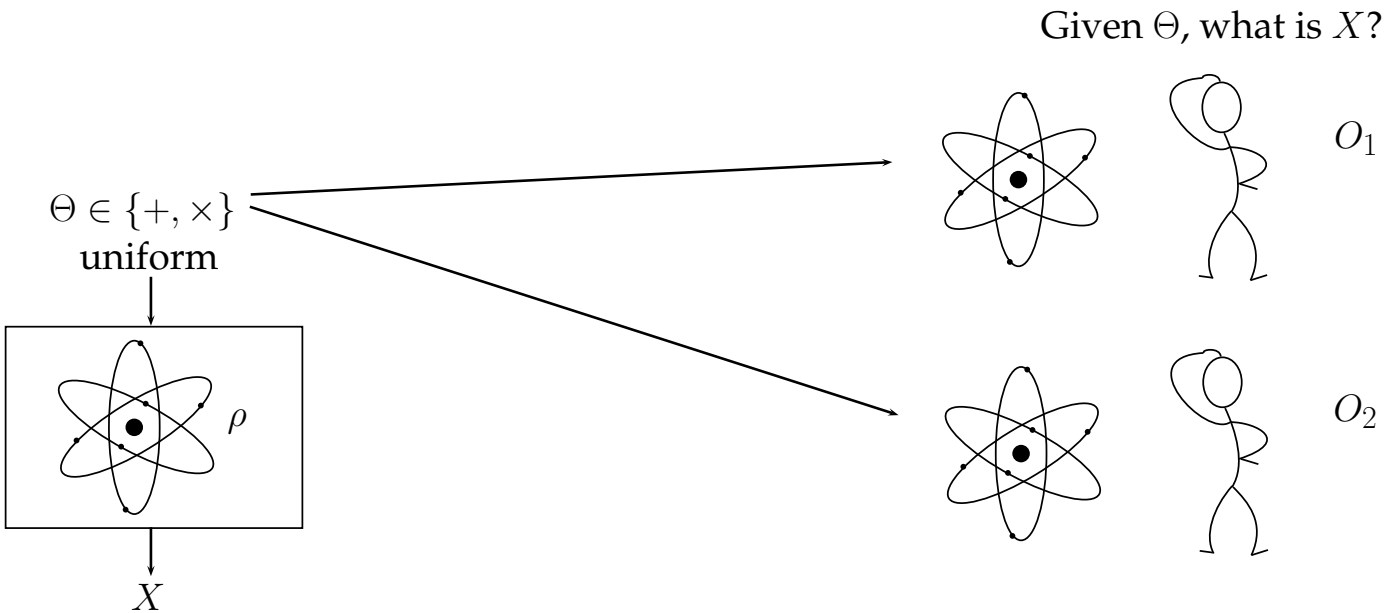
$$H(X|O_1, \Theta) + H(X|O_2, \Theta) \geq \log_2 \frac{1}{c} .$$

- $H(A|B) := H(AB) - H(B)$ is the von Neumann entropy.
- The overlap c is defined as in the classical relation.

Uncertainty and Monogamy of Entanglement



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- Can this be generalized to quantum Rényi entropies, in particular the min-entropy?

Quantum Guessing Probability and the Min-Entropy

- The guessing probability naturally generalizes to the case of classical quantum states $\rho_{XE} = \bigoplus_x P(x)\rho_E^x$:

$$p_{\text{guess}}(X|E) := \sup\left\{ \sum_x P(x)\,\text{tr}(\rho_E^x M_E^x) \,\middle|\, \{M_E^x\} \text{ a POVM} \right\}.$$

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- Then, we retain $H_{\min}(X|E) := -\log p_{\text{guess}}(X|E)$ as proposed by König, Renner & Schaffner [KRS09].
- The operational interpretation remains intact:

Renner [Ren05]: Quantum Leftover Hash Lemma

Let $X \in \mathcal{X}$ be such that $H_{\min}^{\varepsilon}(X|E) \geq k$, and $\ell \in \mathbb{N}$. Then, there exists a family of Hash function $\{f_s\}_s$ from $\mathcal{X}$ to $\{0,1\}^\ell$ such that $f_S(X)$ is δ-close to uniform and independent of E and S, where

$$\delta \sim \sqrt{2^{\ell-k}} + 2\varepsilon.$$

Uncertainty Relation for min-/max Entropies

T & Renner [TR11]: Min-Max Entropic Uncertainty

For any state $\rho_{AO_1O_2}$ and POVMs $\{M_x\}$ and $\{N_x\}$ on A:

$$H_{\min}(X|O_1,\Theta) + H_{\max}(X|O_2,\Theta) \geq \log_2 \frac{1}{c},$$

$$c = \max_{x,y} \left\| \sqrt{M_x}\sqrt{N_y} \right\|_\infty^2 .$$

- Reduces to overlap defined previously for projective measurements.

Uncertainty Relation for min-/max Entropies

T & Renner [TR11]: Smooth Entropic Uncertainty

For any state $\rho_{AO_1O_2}$, $\varepsilon \geq 0$, and POVMs $\{M_x\}$ and $\{N_x\}$ on A:

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- Reduces to overlap defined previously for projective measurements.
- This also generalizes the previous result for the von Neumann entropy due to asymptotic equipartition [TCR09]:

$$\frac{1}{n} H_{\min}^{\varepsilon}(A^n|B^n), \ \frac{1}{n} H_{\max}^{\varepsilon}(A^n|B^n) \xrightarrow{n\to\infty} H(A|B).$$

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- What about general Rényi entropies?

Quantum Rényi Entropy Intermezzo

We want to find a quantum generalization of the Rényi entropy that satisfies the following

- It evaluates to the von Neumann, min-, and max-entropies for $\alpha = 1, \infty, \frac{1}{2}$, respectively.
- It has mathematical properties that we expect from an operational quantity, e.g. a data-processing inequality:

$$H_\alpha(A|B) \leq H_\alpha(A|C) \quad \text{for any quantum channel } \mathcal{E} : B \to C.$$

Such a generalization was recently proposed by Müller-Lennert, Dupuis, Szehr, Fehr & T [MLDS$^+$13] (see also [WWY13]).

Uncertainty Relation for Quantum Rényi Entropy

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Müller-Lennert *et al.* [MLDS$^+$13]: Quantum Rényi Entropy

The quantum conditional Rény entropy is, for $\alpha \geq \frac{1}{2}$,

$$H_\alpha(A|B) := \sup_{\sigma_B} -D_\alpha(\rho_{AB} \| 1_A \otimes \sigma_B),$$

where $D_\alpha(\rho\|\sigma) := \frac{2\alpha}{\alpha-1} \log \left\| \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \right\|_{2\alpha}$ uses the Schatten norm.

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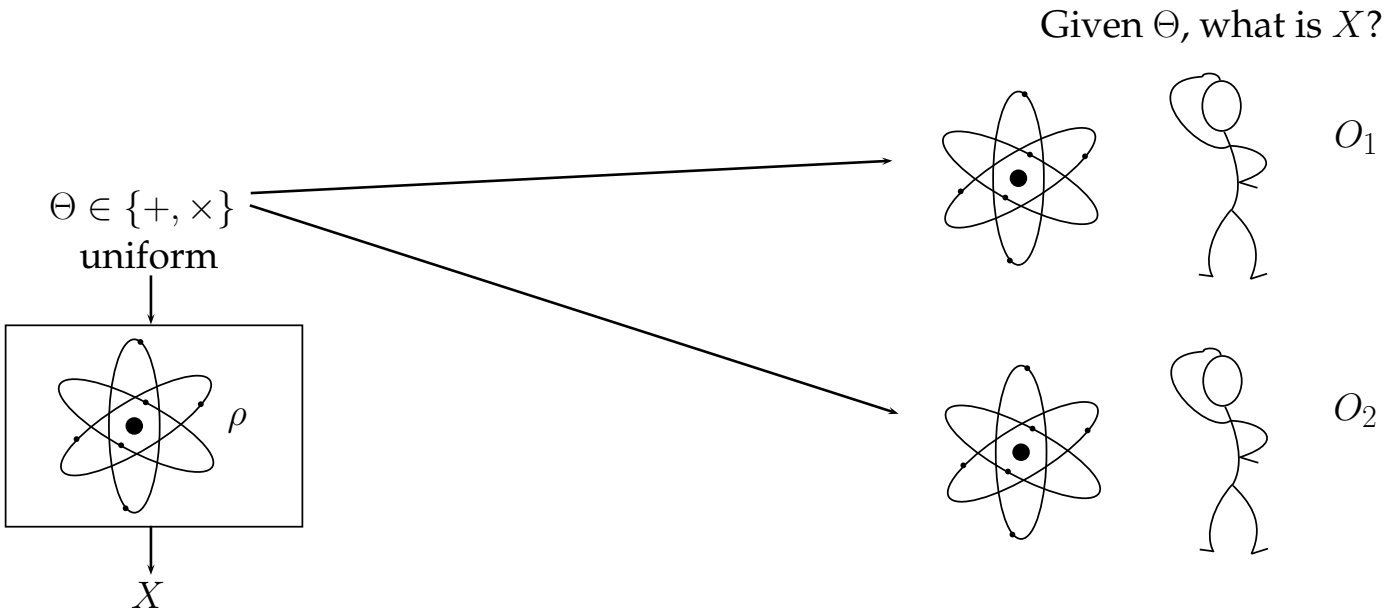
- We recover the well-known special cases for $\alpha = \infty, \frac{1}{2}$:

$$H_{\min}(A|B) = \sup \left\{ \lambda \in \mathbb{R} \,\middle|\, \exists \sigma_B : \rho_{AB} \leq 2^{-\lambda} 1_A \otimes \sigma_B \right\},$$

$$H_{\max}(A|B) = \sup_{\sigma_B} \left(2 \log F(\rho_{AB}, 1_A \otimes \sigma_B) \right).$$

Generalized Masseen & Uffink Relation



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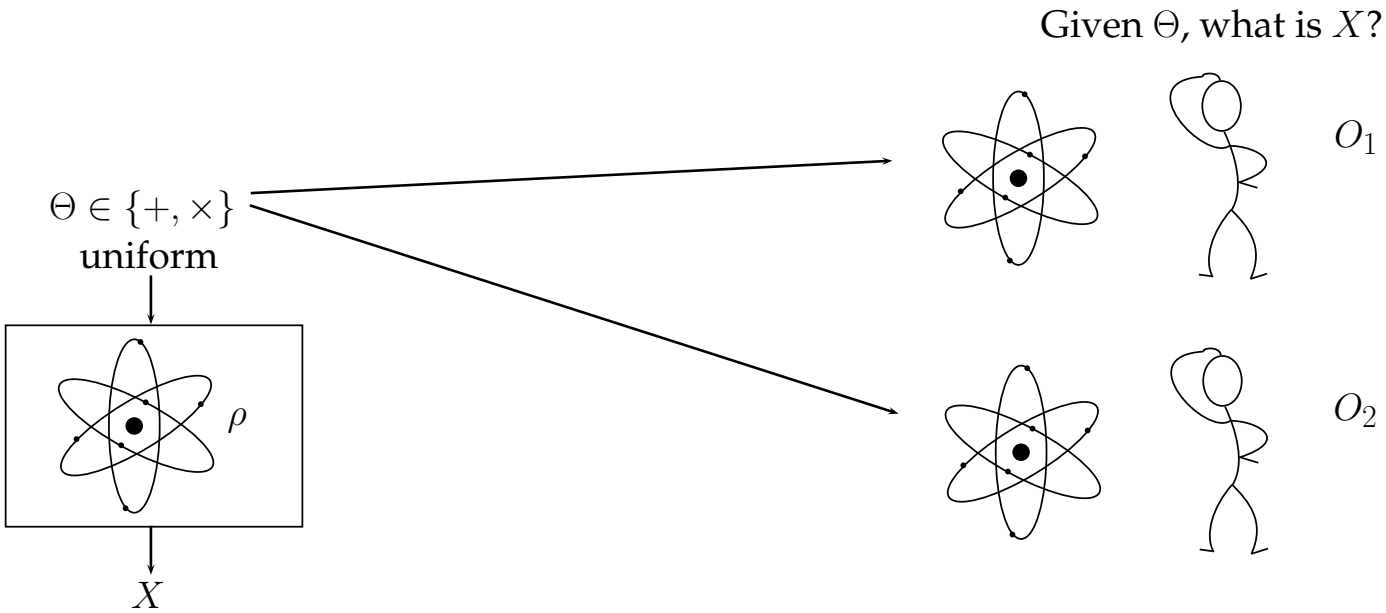
$$\frac{1}{\alpha} + \frac{1}{\beta} = 2, \quad \text{and} \quad c = \max_{x,y} \left\| \sqrt{M_x}\sqrt{N_y} \right\|_\infty^2 .$$

Generalized Masseen & Uffink Relation



Müller-Lennert *et al.* [MLDS+13]: General Entropic Uncertainty

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- Proof only uses a few basic properties of H_α and employs a strategy by Coles, Colbeck, Yu & Zwolak [CCYZ12].

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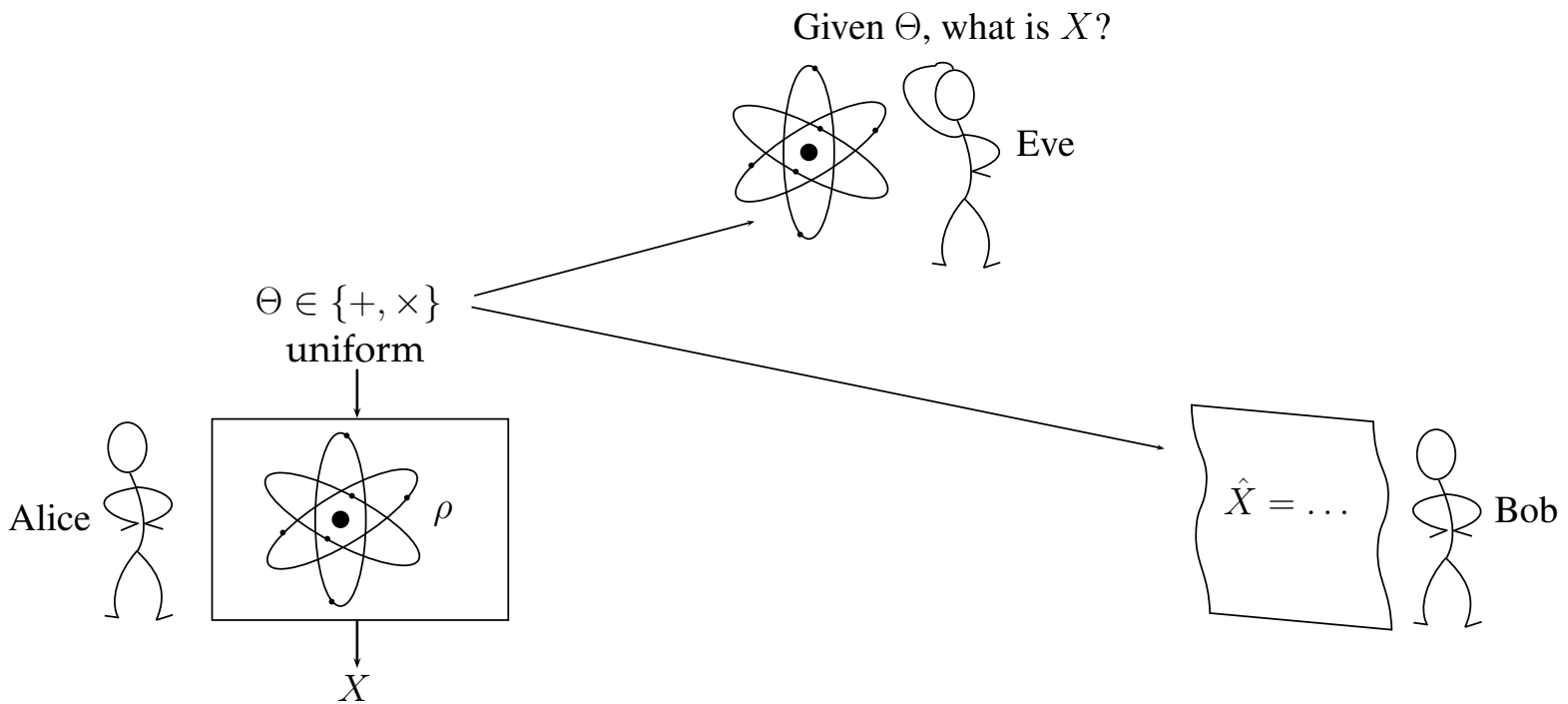

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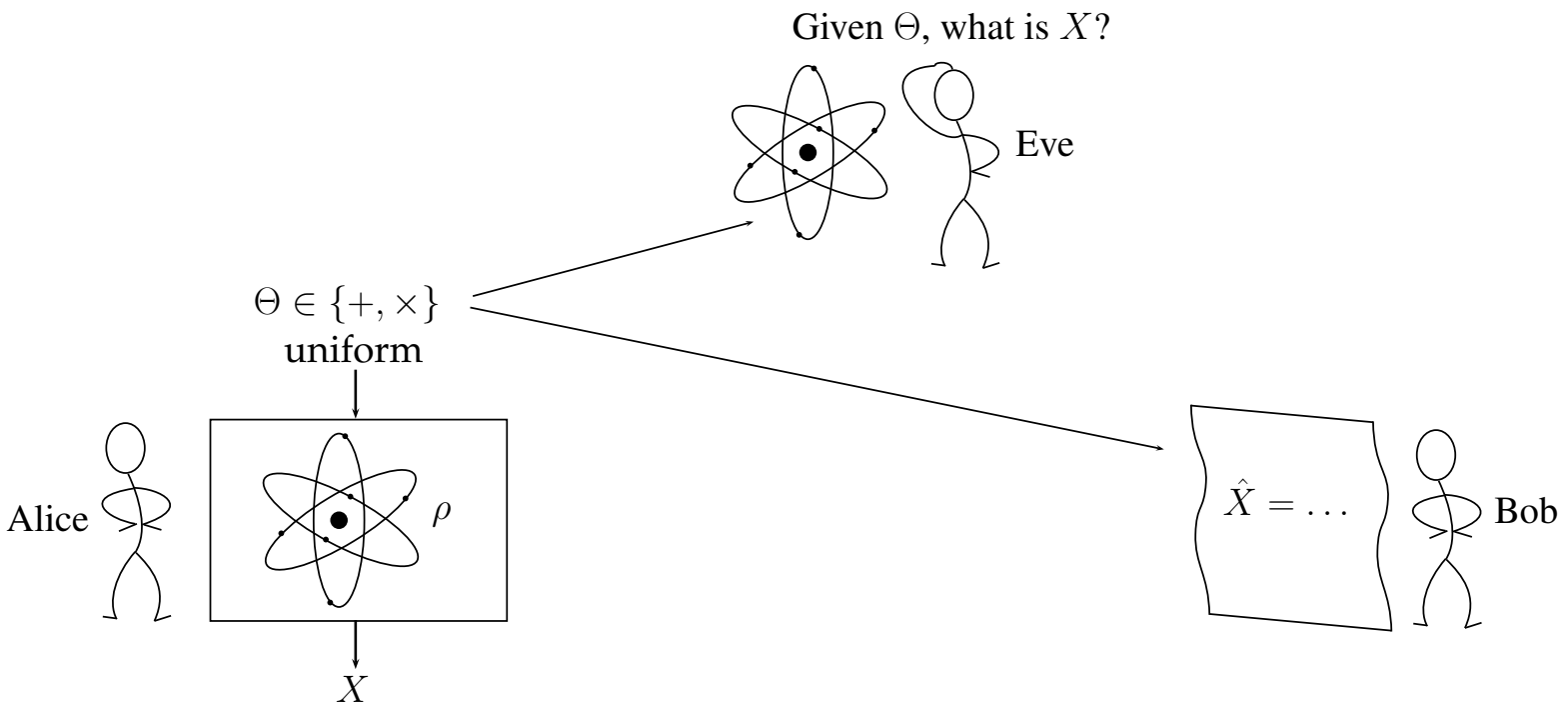
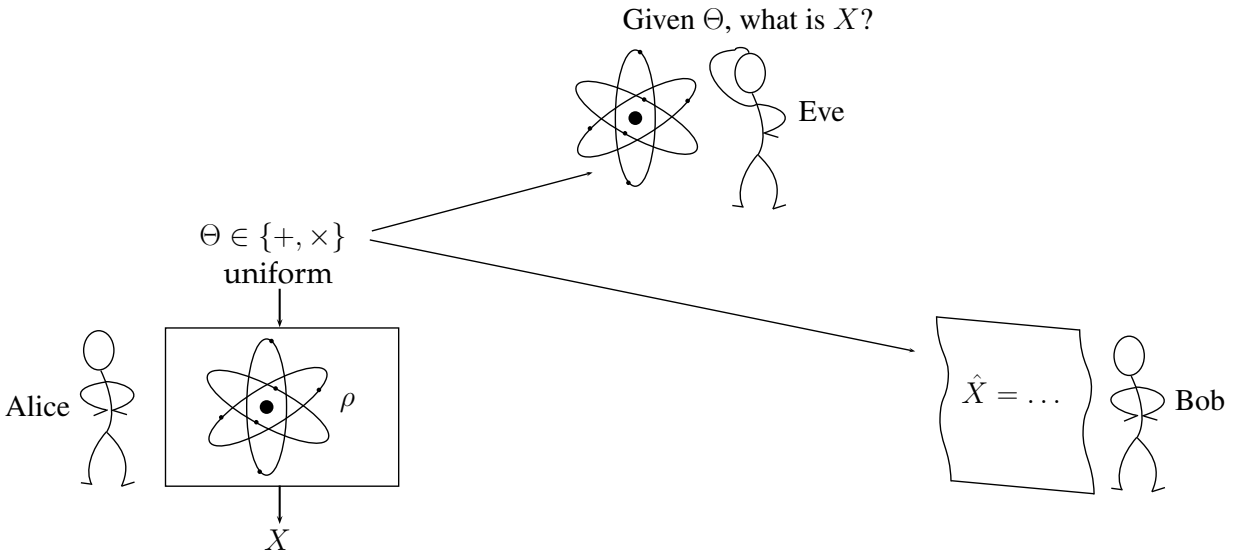


- Here, Θ, X and $\hat{X}$ are n-bit strings.

Uncertainty Relation for n Systems



- Measure either all n systems in '$+$' or in '$\times$'.

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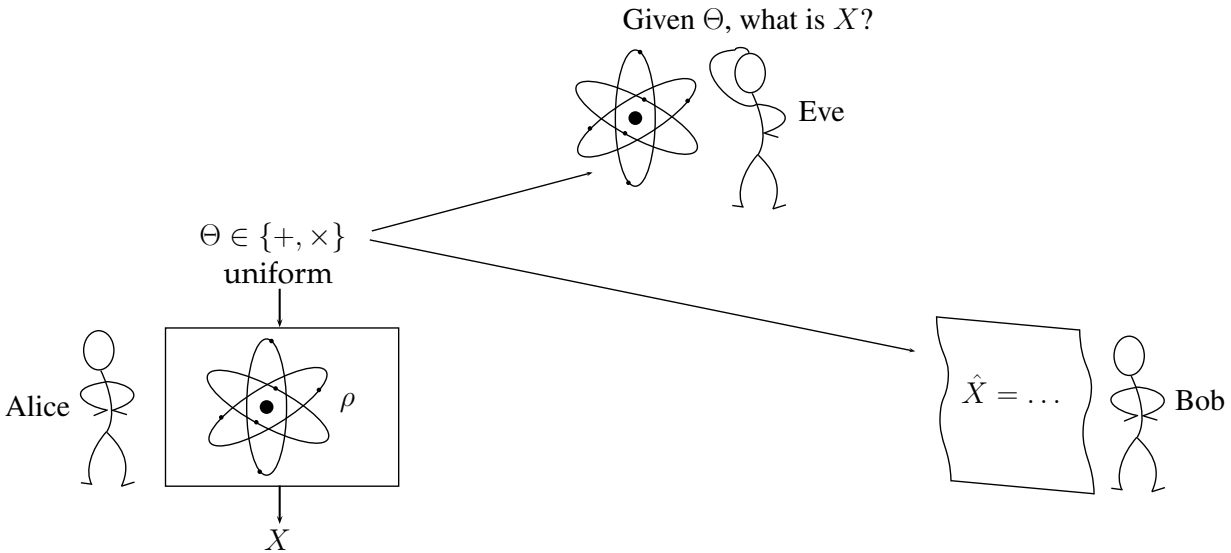

- Measure either all n systems in '$+$' or in '$\times$'.
- Define $F_0 := \{\text{`}++\ldots+\text{'}, \text{`}\times\times\ldots\times\text{'}\}$.

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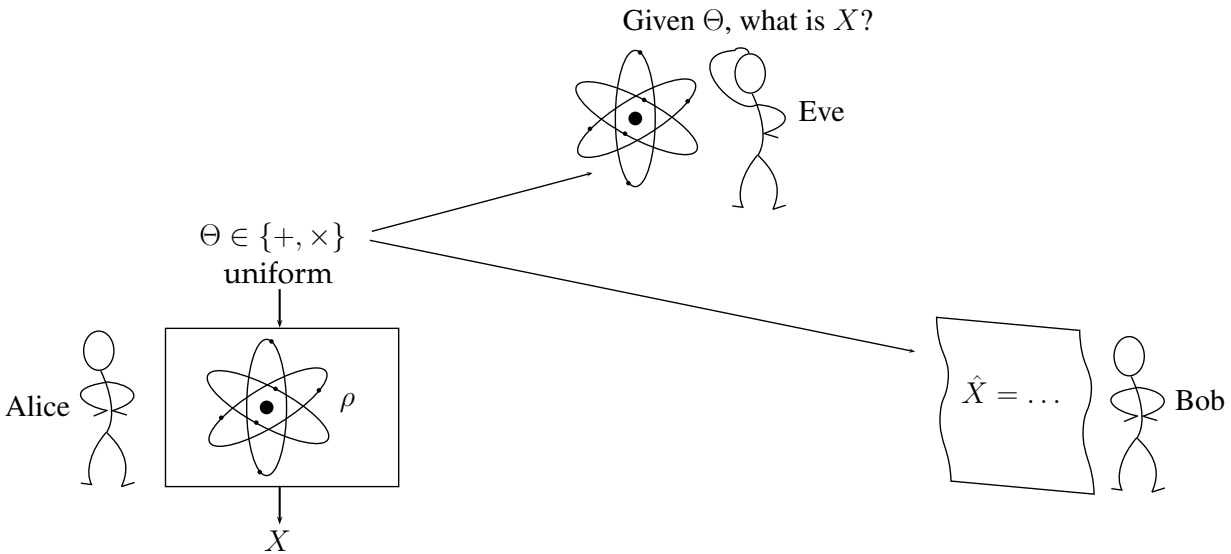

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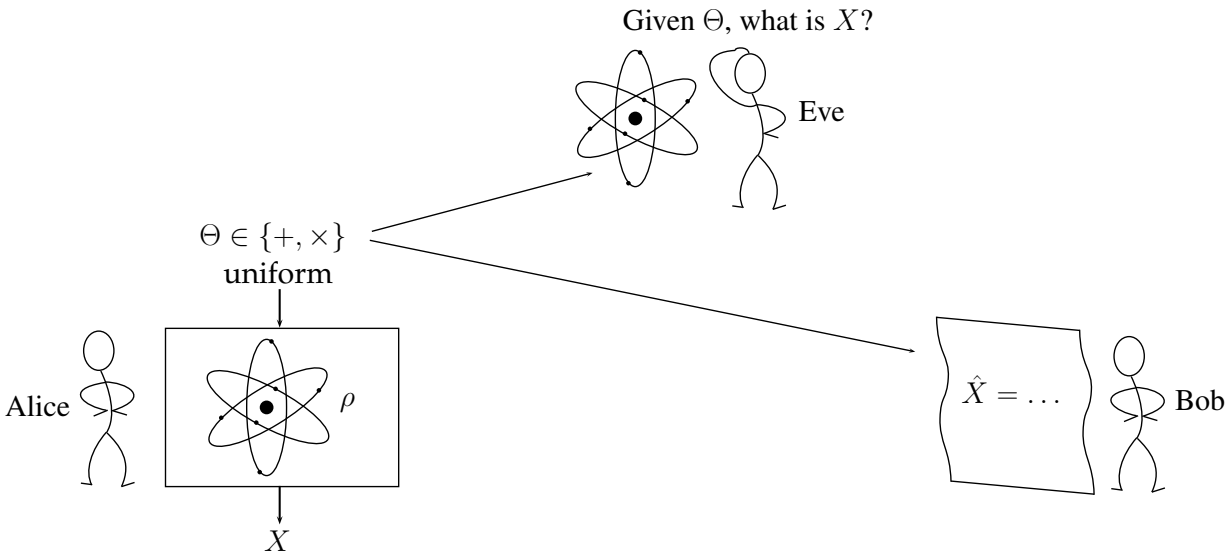

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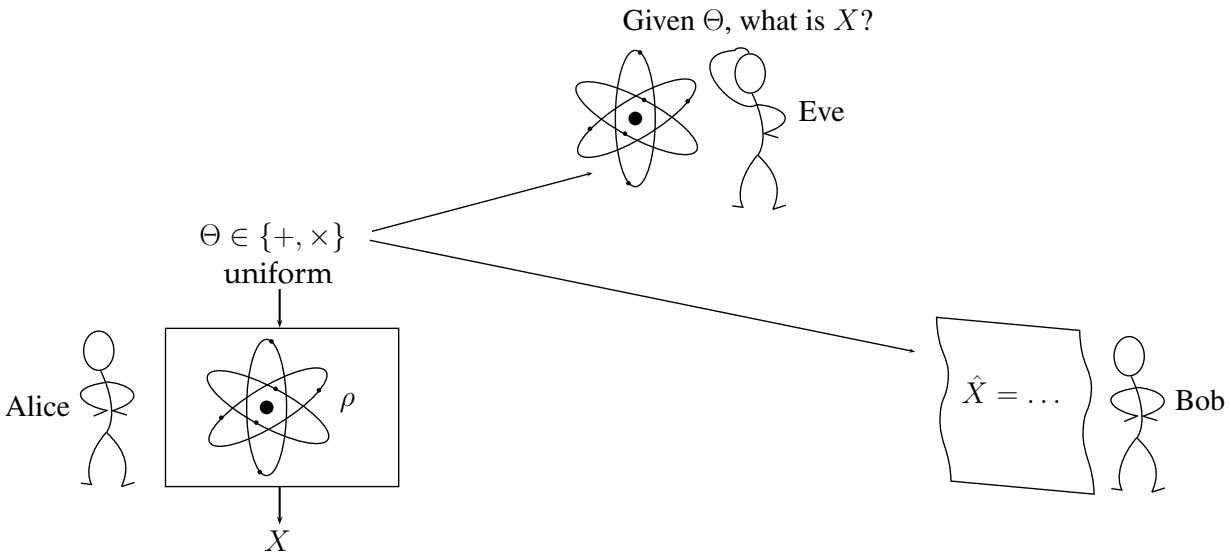
- Analogously for $F_1 := \{\text{`}\times++\ldots+\text{'}, \text{`}+\times\times\ldots\times\text{'}\}$, and so on. Then take the average and use data-processing to get

$$H_{\min}(X|E, \Theta) + H_{\max}(X|\hat{X}) \geq n \log \frac{1}{c}.$$

Security Proof Sketch



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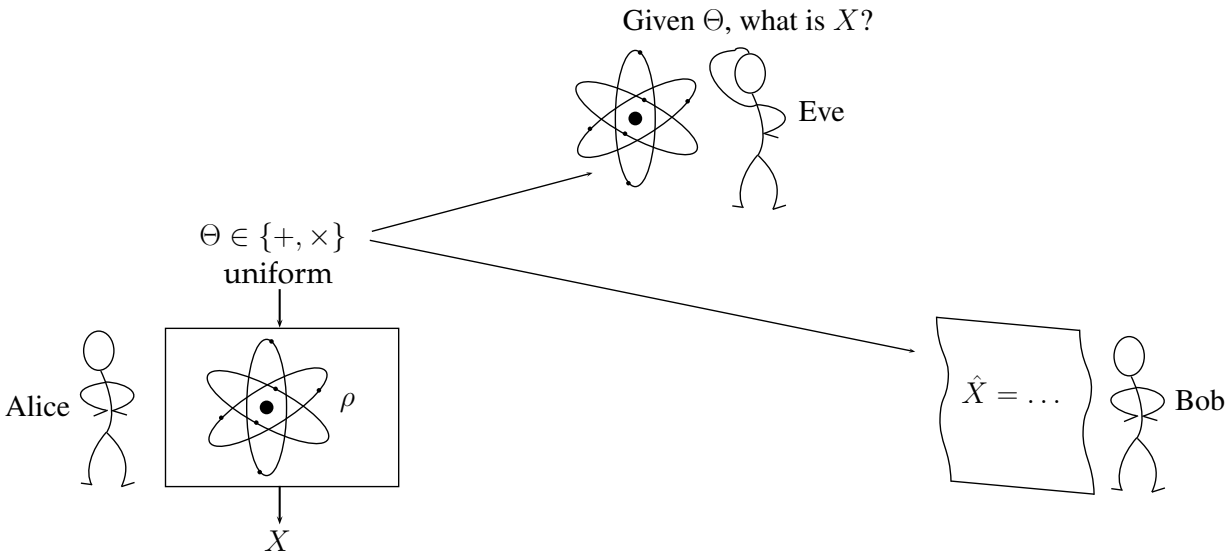

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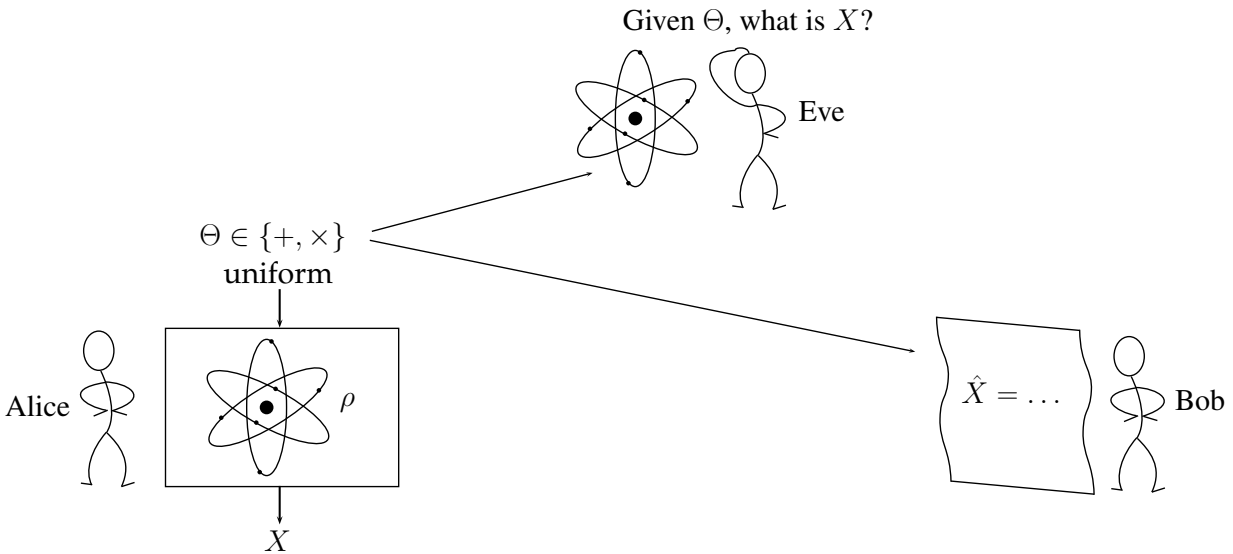

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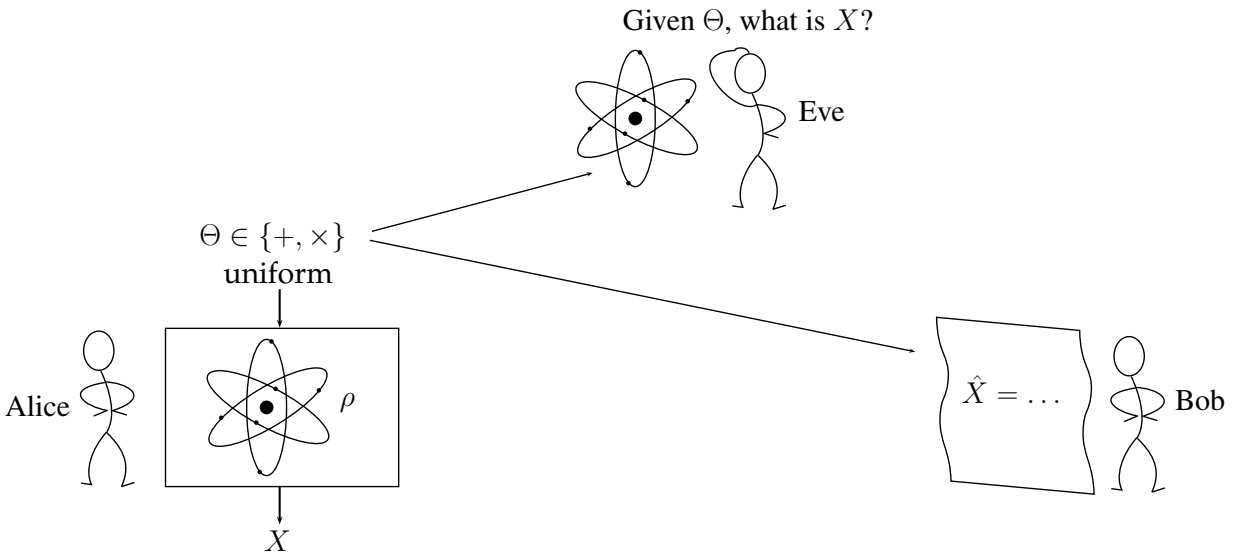

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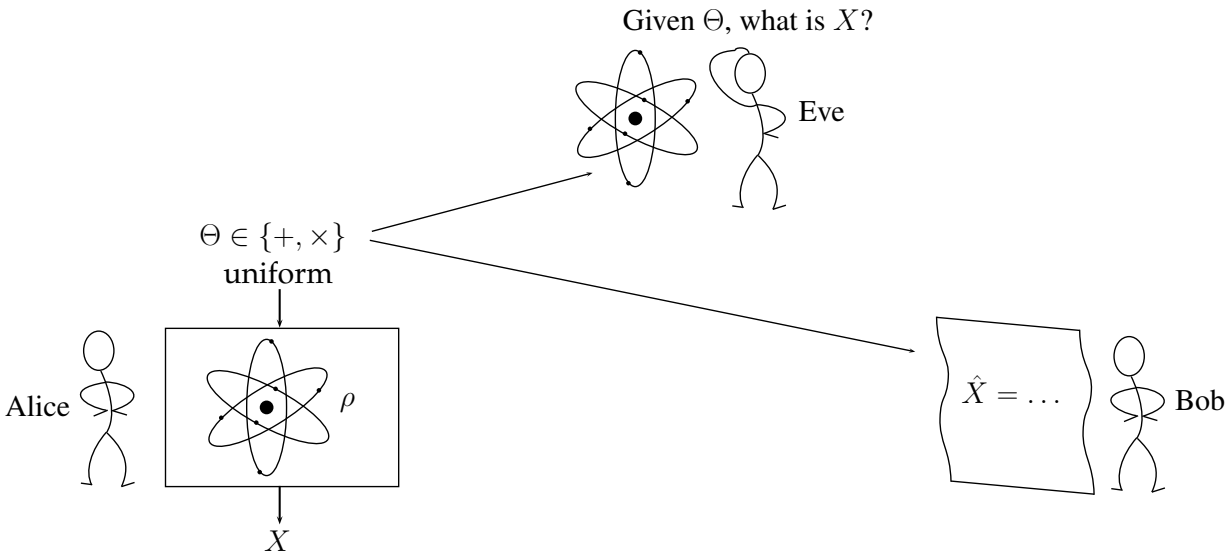

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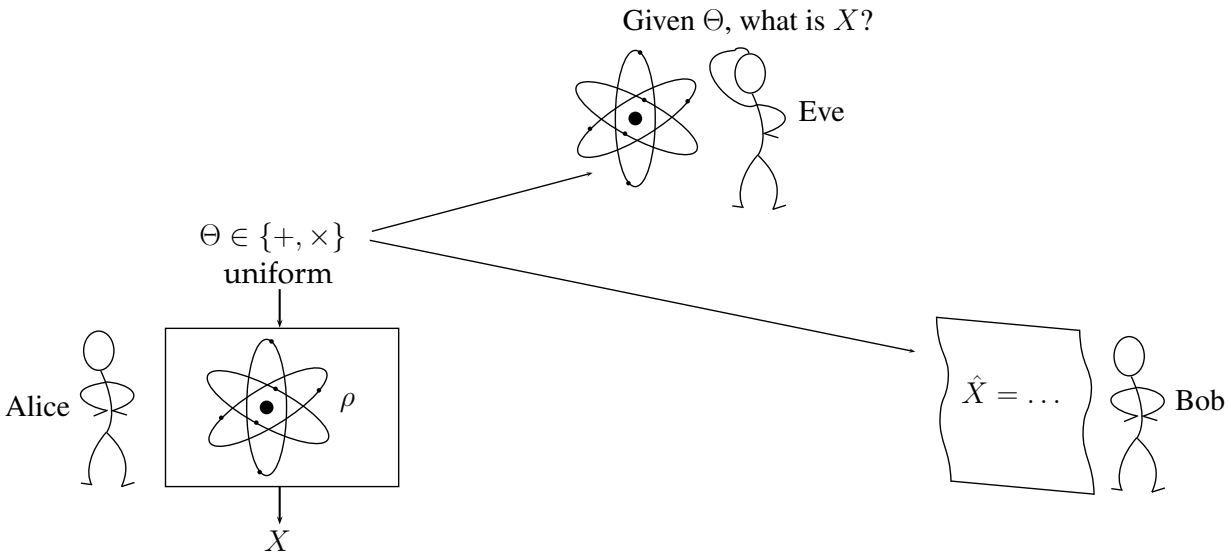

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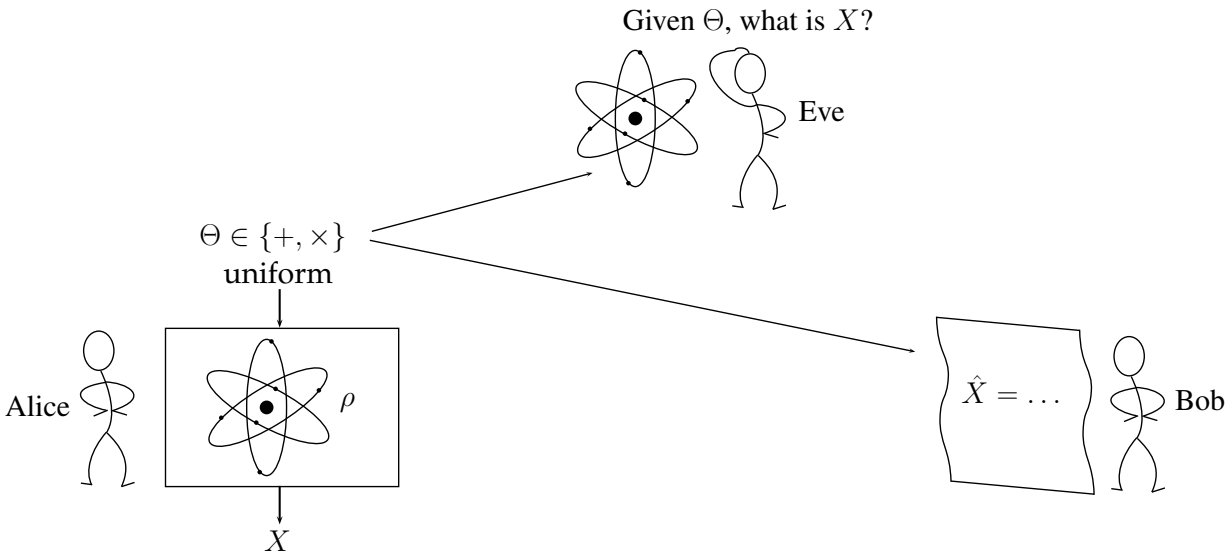

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- Then, estimate $H^{\varepsilon}_{\max}(X|\hat{X}) \lesssim nh(\lambda)$.

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- See also the survey by Wehner & Winter [WW10].

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- This has applications in continuous variable quantum cryptography (CV-QKD). In Furrer *et al.* [FFB+12], we show security for a protocol based on squeezed Gaussian states.
  - In the finite key length regime and for general attacks!
  - Previous security proofs of CV-QKD only provided security under unrealistic assumptions on the eavesdropper.

Take Home Message

Entropic uncertainty relations allow us to formalize physical intuition in information theoretic terms.

$+$

The min-entropy is the preferred measure of uncertainty for cryptographic applications.

$=$

Uncertainty relations in terms of the min-entropy consequently have a wide range of applications in quantum cryptography.

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